Total and Annular Solar Eclipses

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1 Fundamental Plane

Consider Fig. 1. Let E be the center of the Earth, and let all other points in the diagram lie on the surface of a sphere whose center is E. EC is a straight line that is parallel to the straight line passing through the apparent (i.e., as seen from the Earth, taking aberration into account) centers of the Sun and the Moon. DBA is a plane, perpendicular to EC, that passes through E. This plane is known as the *fundamental plane*. Let $FCPB$ be the arc of a great circle, centered on E, and let EP be directed toward the northern celestial pole. Let the plane $D\Upsilon FYHA$ coincide with the Earth's equatorial plane, let $E\Upsilon$ be directed towards the vernal equinox, and let the arc of the great circle PH lie in the plane of the Greenwich meridian. Finally, let X be a celestial body (either the Sun or the Moon), and let PXY, CX, and AX be the arcs of great circles.

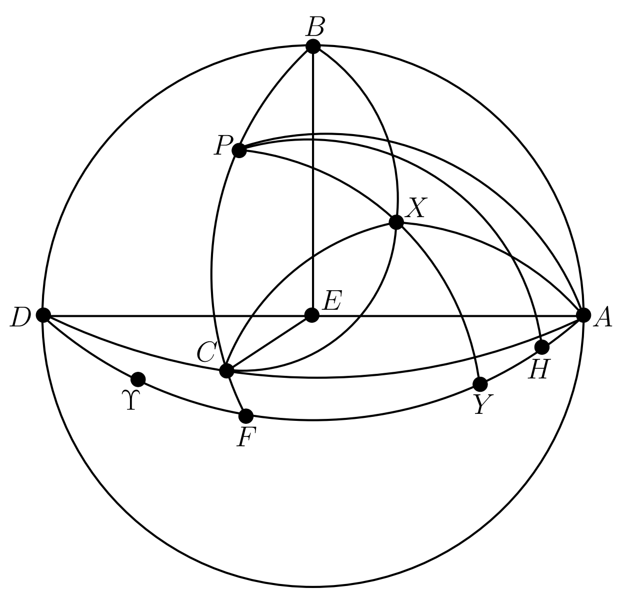

Figure 1: The fundamental plane.

All arcs referred to in the following are the arcs of great circles. It is clear, from the previous description, that the arcs FA, CB, PA, PH, PY, and PF are all 90°. Let d and a denote the declination and right ascension of point C. It follows that arc CF is d, arc PC is $90^\circ - d$, and arc PB is d. Furthermore, arc ΥF is a. Let δ and α denote the apparent declination and right ascension of X, respectively. It follows that arc XY is δ, and arc XP is $90^\circ - \delta$. Furthermore, arc ΥY is α, arc FY is $\alpha - a$, and arc YA is $90^\circ - \alpha + a$. Now, the spherical angles APB and APC are both 90°. However, the spherical angle XPA is $90^\circ - \alpha + a$ (because the arc YA, whose pole is P, is $90^\circ - \alpha + a$). Hence, the spherical angles XPB and XPC are $180^\circ - \alpha + a$ and $\alpha - a$, respectively.

Let us define a right-handed Cartesian coordinate system, x, y, z, whose origin is E, and which is such that the x-axis corresponds to EA, the y-axis to EB, and the z-axis to EC. These axes are referred to as *fundamental axes*, and the corresponding Cartesian coordinate system is known as the *fundamental system*. Let X correspond to the center of the Sun, and let x, y, z be its coordinates in the fundamental system. It follows that

$$x = r\,\cos AX, \tag{1}$$

$$y = r\,\cos BX, \tag{2}$$

$$z = r\,\cos CX, \tag{3}$$

where r is the Sun's apparent geocentric distance. Note that δ and α now refer to the apparent declination and right ascension of the Sun, respectively.

Now, in the spherical triangle XPA, sides PA and XP are 90° and $90^\circ - \delta$, respectively, whereas angle XPA is $90^\circ - \alpha + a$. Hence,

$$\begin{aligned}\cos AX &= \cos PA\,\cos XP + \sin PA\,\sin XP\,\cos XPA \\ &= \cos\delta\,\sin(\alpha - a).\end{aligned} \tag{4}$$

In the spherical triangle XPB, sides PB and XP are d and $90^\circ - \delta$, respectively, whereas angle XPB is $180^\circ - \alpha + a$. Hence,

$$\begin{aligned}\cos BX &= \cos PB\,\cos XP + \sin PB\,\sin XP\,\cos XPB \\ &= \cos d\,\sin\delta - \sin d\,\cos\delta\,\cos(\alpha - a).\end{aligned} \tag{5}$$

Finally, in the spherical triangle XPC, sides PC and XP are $90^\circ - d$ and $90^\circ - \delta$, respectively, whereas angle XPC is $\alpha - a$. Hence,

$$\begin{aligned}\cos CX &= \cos PC\,\cos XP + \sin PC\,\sin XP\,\cos XPC \\ &= \sin d\,\sin\delta + \cos d\,\cos\delta\,\cos(\alpha - a).\end{aligned} \tag{6}$$

It follows that the Cartesian coordinates of the Sun in the fundamental system are

$$x = r\,\cos\delta\,\sin(\alpha - a), \tag{7}$$

$$y = r\,[\sin\delta\,\cos d - \cos\delta\,\sin d\,\cos(\alpha - a)]\,, \tag{8}$$

$$z = r\,[\sin\delta\,\sin d + \cos\delta\,\cos d\,\cos(\alpha - a)]\,. \tag{9}$$

Let x_1, y_1, z_1 be the Cartesian coordinates of the center of the Moon in the fundamental system. By analogy with the previous analysis, we can write

$$x_1 = r_1 \cos\delta_1 \sin(\alpha_1 - a), \tag{10}$$

$$y_1 = r_1 [\sin\delta_1 \cos d - \cos\delta_1 \sin d \cos(\alpha_1 - a)], \tag{11}$$

$$z_1 = r_1 [\sin\delta_1 \sin d + \cos\delta_1 \cos d \cos(\alpha_1 - a)], \tag{12}$$

where r_1 is the Moon's apparent geocentric distance, whereas δ_1 and α_1 are its apparent declination and right ascension, respectively.

Now, because the z-axis is parallel to the straight line joining the centers of the Sun and the Moon, it follows that

$$x = x_1, \tag{13}$$

$$y = y_1. \tag{14}$$

Furthermore, (x_1, y_1) are the coordinates of the center of the shadow of the Moon cast by the Sun on the fundamental plane.

Equations (7), (10), and (13) yield

$$r \cos\delta \sin(\alpha - a) = r_1 \cos\delta_1 \sin(\alpha_1 - a), \tag{15}$$

or

$$r \cos\delta (\sin\alpha \cos a - \cos\alpha \sin a) = r_1 \cos\delta_1 (\sin\alpha_1 \cos a - \cos\alpha_1 \sin a), \tag{16}$$

which can be rearranged to give

$$\frac{\sin a}{\cos a} = \frac{r \cos\delta \sin\alpha - r_1 \cos\delta_1 \sin\alpha_1}{r \cos\delta \cos\alpha - r_1 \cos\delta_1 \cos\alpha_1}. \tag{17}$$

Equations (8), (11), and (14) yield

$$r [\sin\delta \cos d - \cos\delta \sin d \cos(\alpha - a)] = r_1 [\sin\delta_1 \cos d - \cos\delta_1 \sin d \cos(\alpha_1 - a)], \tag{18}$$

which can be rearranged to give

$$\tan d = \frac{r \sin\delta - r_1 \sin\delta_1}{r \cos\delta \cos(\alpha - a) - r_1 \cos\delta_1 \cos(\alpha_1 - a)}. \tag{19}$$

If we define

$$X = r \cos\delta \cos\alpha - r_1 \cos\delta_1 \cos\alpha_1, \tag{20}$$

$$Y = r \cos\delta \sin\alpha - r_1 \cos\delta_1 \sin\alpha_1, \tag{21}$$

$$Z = r \sin\delta - r_1 \sin\delta_1, \tag{22}$$

then it is clear from Eq. (17) that

$$\cos a = \frac{X}{(X^2 + Y^2)^{1/2}}, \tag{23}$$

$$\sin a = \frac{Y}{(X^2 + Y^2)^{1/2}}. \tag{24}$$

Furthermore,

$$r \cos\delta \cos(\alpha - a) - r_1 \cos\delta_1 \cos(\alpha_1 - a) = X \cos a + Y \sin a = \left(X^2 + Y^2\right)^{1/2}, \tag{25}$$

Hence, Eq. (19) yields

$$\tan d = \frac{Z}{(X^2 + Y^2)^{1/2}}. \tag{26}$$

It follows that

$$\frac{\sin a}{\cos a} = \frac{Y}{X}, \tag{27}$$

$$\sin d = \frac{Z}{(X^2 + Y^2 + Z^2)^{1/2}}. \tag{28}$$

Equations (27) and (28) enable the right ascension and declination of point C to be calculated directly from the apparent positions of the Sun and the Moon.

Let G denote Greenwich apparent sidereal time. It follows that, in Fig. 1, the arc $H\Upsilon$ is G (converted into an angle). Moreover, arc $F\Upsilon$ is a. Let μ denote the hour angle of point C at the Greenwich meridian. It follows that arc HF is μ. Hence,

$$\mu = G - a. \tag{29}$$

2 Geometry of Moon's Shadow in Fundamental Plane

Referring to Fig. 2, ASA' represents the Sun, whose center is at S. Likewise, BMB' represents the Moon, whose center is at M. $BTRV_2B'B$ is the Moon's umbra, whose vertex is V_2. Likewise, $BHCDKB'B$ is the Moon's penumbra, whose virtual vertex is V_1. The fundamental plane corresponds $CRFD$, and F is the center of the Moon's shadow in this plane.

Let f_1 be the half-angle of the penumbral cone. It follows that angles AV_1S and MV_1B are both equal to f_1. Moreover, angles SAV_1 and MBV_1 are both 90° (since AV_1 and BV_1 are tangents to the Sun and Moon, respectively). Simple trigonometry reveals that

$$\sin f_1 = \frac{R}{SV_1} = \frac{k}{V_1M} = \frac{R + k}{SM}, \tag{30}$$

where R is the radius of the Sun, and k the radius of the Moon. Now,

$$SM = z - z_1. \tag{31}$$

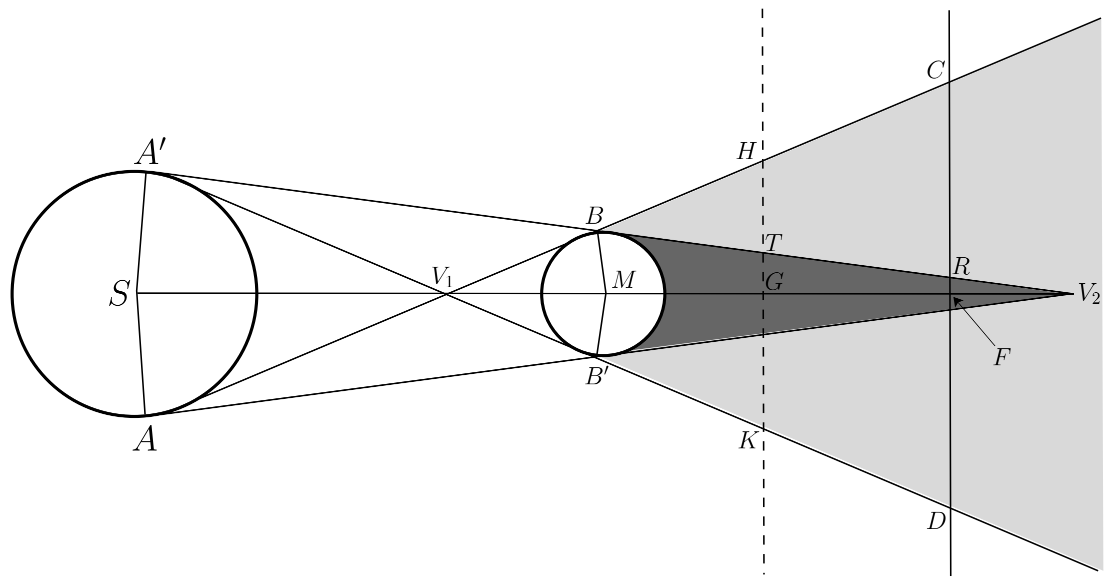

Figure 2: The Moon's shadow.

Hence, we obtain

$$\sin f_1 = \frac{R+k}{z-z_1}. \tag{32}$$

Let f_2 be the half-angle of the umbral cone. It follows that angles $A'V_2S$ and BV_2M are both equal to f_2. Moreover, angles $SA'V_2$ and MBV_2 are both 90° (since $A'V_2$ and BV_2 are tangents to the Sun and Moon, respectively). Simple trigonometry reveals that

$$\sin f_2 = \frac{R}{SV_2} = \frac{k}{V_2M} = \frac{R-k}{SM}. \tag{33}$$

Hence,

$$\sin f_2 = \frac{R-k}{z-z_1}. \tag{34}$$

Again referring to Fig. 2, $FM = z_1$, where z_1 is the Moon's z-coordinate in the fundamental system, and $V_1M = k/\sin f_1$. Let us denote the distance FV_1, which is the z-coordinate of the virtual vertex of the penumbral cone, as c_1. It follows that

$$c_1 = z_1 + \frac{k}{\sin f_1}. \tag{35}$$

Now, $V_2M = k/\sin f_2$. Let us denote the distance $-FV_2$, which is the z-coordinate of the vertex of the umbral cone as c_2. It follows that

$$c_2 = z_1 - \frac{k}{\sin f_2}. \tag{36}$$

Note that c_2 is positive when the vertex lies in front (i.e., on the positive z side) of the fundamental plane, and vice versa.

Let $l_1 = FC$ be the radius of the intersection of the penumbral cone with the fundamental plane. It follows that

$$l_1 = c_1 \tan f_1, \tag{37}$$

which implies that

$$l_1 = z_1 \tan f_1 + \frac{k}{\cos f_1}. \tag{38}$$

Let $l_2 = -FR$ be minus the radius of the intersection of the umbral cone with the fundamental plane. It follows that

$$l_2 = c_2 \tan f_2, \tag{39}$$

which implies that

$$l_2 = z_1 \tan f_2 - \frac{k}{\cos f_2}. \tag{40}$$

Note that l_1 is always positive. On the other hand, l_2 is positive when the vertex of the umbral cone lies in front of the fundamental plane, and vice versa.

The quantities x, y, d, μ, f_1, f_2, l_1, and l_2 are known as the *Besselian elements* of the eclipse.

3 Geometry of Moon's Shadow in Observation Plane

Consider an observer on the surface of the Earth. Let the observer's coordinates in the fundamental system be ξ, η, ζ. Let the observer lie somewhere in the plane $KGTH$, shown in Fig. 2, which is parallel to the fundamental plane, and is known as the *observation plane*. Here, point G is the center of the Moon's shadow in the observation plane. It follows that the equation of the observation plane is $z = \zeta$.

Let $L_1 = GH$ be the radius of the intersection of the penumbral cone with the observation plane. Now, $FG = \zeta$ and $FV_1 = c_1$, so $GV_1 = c_1 - \zeta$, and

$$L_1 = (c_1 - \zeta) \tan f_1, \tag{41}$$

which implies that

$$L_1 = l_1 - \zeta \tan f_1. \tag{42}$$

Let $L_2 = -GT$ be minus the radius of the intersection of the umbral cone with the observation plane. Now, $FG = \zeta$ and $FV_2 = -c_2$, so $GV_2 = \zeta - c_2$, and

$$L_2 = -(\zeta - c_2) \tan f_2, \tag{43}$$

which implies that

$$L_2 = l_2 - \zeta \tan f_2. \tag{44}$$

Note that L_1 is always positive. On the other hand, L_2 is negative when the vertex of the umbral cone lies behind the observation plane, and vice versa. This implies that the eclipse is total when $L_2 < 0$, and annular when $L_2 > 0$.

4 Observer Coordinates

Let us reinterpret Fig. 1. Now, all points except E (which again represents the center of the Earth) lie on the surface of the Earth. As before, EC is parallel to the straight line joining the apparent centers of the Sun and the Moon, DBA is the fundamental plane, $E\Upsilon$ is directed toward the vernal equinox, and PH lies in the plane of the Greenwich meridian. However, P is now the Earth's northern geographic pole, $D\Upsilon F\,Y\,HA$ the Earth's equator, and X the position of the observer. As before, the arc HF is equal to μ. Let ϕ' and λ be the observer's geocentric latitude and longitude (measured westward from the Greenwich meridian), respectively. It follows that arc XY is ϕ', and arc XP is $90^\circ - \phi'$. Moreover, arc HY is λ, which implies that arc YF is $\mu - \lambda$. Let

$$h = \mu - \lambda. \tag{45}$$

It follows that the spherical angle XPC is h, whereas the spherical angle XPA is $90^\circ - h$, and, finally, the spherical angle XPB is $180^\circ - h$.

The Cartesian components of the observer in the fundamental system are written

$$\xi = \rho\, \cos AX, \tag{46}$$

$$\eta = \rho\, \cos BX, \tag{47}$$

$$\zeta = \rho\, \cos CX, \tag{48}$$

where ρ is the observer's geocentric distance.

Now, in the spherical triangle XPA, sides PA and XP are 90° and $90^\circ - \phi'$, respectively, whereas angle XPA is $90^\circ - h$. Hence,

$$\begin{aligned}\cos AX &= \cos PA\, \cos XP + \sin PA\, \sin XP\, \cos XPA \\ &= \cos\phi'\, \sin h. \end{aligned} \tag{49}$$

In the spherical triangle XPB, sides PB and XP are d and $90^\circ - \phi'$, respectively, whereas angle XPB is $180^\circ - h$. Hence,

$$\begin{aligned}\cos BX &= \cos PB\, \cos XP + \sin PB\, \sin XP\, \cos XPB \\ &= \cos d\, \sin\phi' - \sin d\, \cos\phi'\, \cos h. \end{aligned} \tag{50}$$

Finally, in the spherical triangle XPC, sides PC and XP are $90^\circ - d$ and $90^\circ - \phi'$, respectively, whereas angle XPC is h. Hence,

$$\begin{aligned}\cos CX &= \cos PC\, \cos XP + \sin PC\, \sin XP\, \cos XPC \\ &= \sin d\, \sin\phi' + \cos d\, \cos\phi'\, \cos h. \end{aligned} \tag{51}$$

It follows that the Cartesian coordinates of the observer in the fundamental system are

$$\xi = \rho\, \cos\phi'\, \sin h, \tag{52}$$

$$\eta = \rho\left(\sin\phi'\, \cos d - \cos\phi'\, \sin d\, \cos h\right), \tag{53}$$

$$\zeta = \rho\left(\sin\phi'\, \sin d + \cos\phi'\, \cos d\, \cos h\right). \tag{54}$$

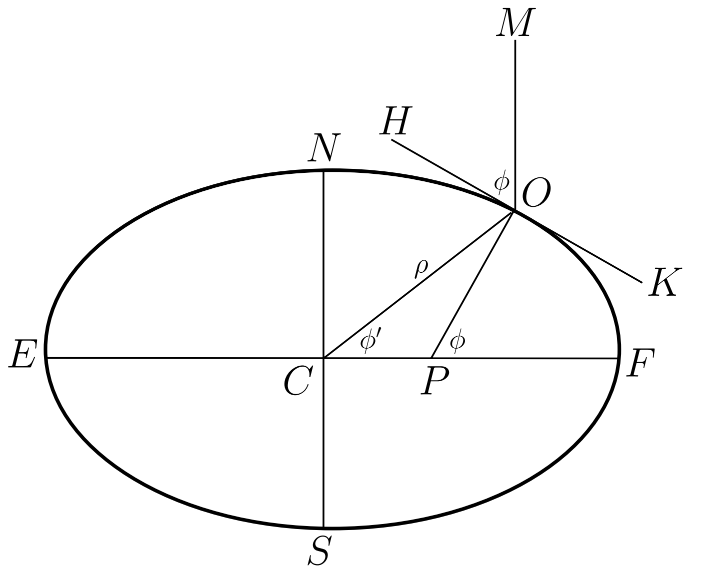

Figure 3: The figure of the Earth.

5 Figure of Earth

Figure 3 represents a meridian cross-section of the Earth. C is the Earth's center, N its north pole, and S its south pole. The straight line ECF is perpendicular to the straight line NCS, HK is the horizontal plane of the observer, O, and OP is perpendicular to HK. The line OM, parallel to SN, subtends an angle with OH that is the altitude of the northern celestial pole, and is, by definition, the observer's *geographic latitude*, ϕ. Angle OPF is also equal to ϕ. The radius vector OC, which is of magnitude ρ, joins the observer to the center of the Earth, and subtends an angle with CF that, by definition, is the observer's *geocentric latitude*, ϕ'.

Let $a = CF$ be the Earth's equatorial radius, and $b = NC$ the Earth's polar radius. We can write

$$\frac{b}{a} = 1 - f, \tag{55}$$

where

$$f = \frac{1}{298.257} \tag{56}$$

is the Earth's flattening factor.

Let us define Cartesian coordinates such that CF is the X-axis, and CN the Z-axis. Assuming that the Earth is spheroidal, the equation of its surface is

$$\frac{X^2}{a^2} + \frac{Z^2}{b^2} = 1. \tag{57}$$

Thus, at point O,

$$\tan\phi' = \frac{Z}{X}. \tag{58}$$

Recall that the slope of the tangent line at a point is dZ/dX, and that the slope of the normal line at the same point is $-dX/dZ$. Hence,

$$\tan\phi = -\frac{dX}{dZ}. \tag{59}$$

However, according to Eqs. (57) and (58),

$$\frac{dX}{dZ} = -\frac{a^2}{b^2}\,\frac{Z}{X} = -\frac{a^2}{b^2}\,\tan\phi'. \tag{60}$$

Hence,

$$\tan\phi' = (1-f)^2\,\tan\phi, \tag{61}$$

where use has been made of Eq. (55).

The equation of the Earth's surface can also be written in parametric form:

$$X = a\,\cos u, \tag{62}$$

$$Z = b\,\sin u. \tag{63}$$

It follows that

$$\tan\phi' = \frac{Z}{X} = \frac{b}{a}\,\tan u. \tag{64}$$

Hence,

$$\tan\phi' = (1-f)\,\tan u, \tag{65}$$

$$\tan\phi = \frac{\tan u}{1-f}, \tag{66}$$

and

$$X \equiv \rho\,\cos\phi' = a\,\cos u, \tag{67}$$

$$Z \equiv \rho\,\sin\phi' = b\,\sin u = a\,(1-f)\,\sin u, \tag{68}$$

where ρ is the observer's geocentric distance.

Finally, it follows from Eqs. (52)–(54) and (67)–(68) that the observer's coordinates in the fundamental system can be written

$$\xi = \cos u\,\sin h, \tag{69}$$

$$\eta = (1-f)\,\sin u\cos d - \cos u\,\sin d\,\cos h, \tag{70}$$

$$\zeta = (1-f)\,\sin u\,\sin d + \cos u\,\cos d\,\cos h. \tag{71}$$

Here, and in the following, all lengths are expressed as fractions of the Earth's equatorial radius,

$$a = 6378.14\,\text{km}. \tag{72}$$

6 Determination of Observer Latitude and Longitude

Consider an observer on the near (to the Sun and Moon) side of the Earth's surface whose coordinates in the fundamental system are (ξ, η, ζ). It follows from Eqs. (69) and (70) that

$$\xi = \cos u \, \sin h, \tag{73}$$

$$\eta = (1 - f) \, \sin u \, \cos d - \cos u \, \sin d \, \cos h. \tag{74}$$

Eliminating h between the previous two equations, we obtain

$$[\eta - (1 - f) \, \sin u \, \cos d]^2 = \cos^2 u \, \sin^2 d - \xi^2 \, \sin^2 d. \tag{75}$$

It is convenient to make the following definitions:

$$\omega = \left[1 - f\,(2 - f) \, \cos^2 d\right]^{-1/2}, \tag{76}$$

$$\eta_1 = \omega \, \eta, \tag{77}$$

$$b_1 = \omega \, \sin d, \tag{78}$$

$$b_2 = (1 - f) \, \omega \, \cos d. \tag{79}$$

Note that $b_1^2 + b_2^2 = 1$. The previous five equations can be combined to give

$$\sin^2 u - 2 \, b_2 \, \eta_1 \, \sin u + \eta_1^2 + b_1^2 \, (\xi^2 - 1) = 0. \tag{80}$$

It follows that

$$\sin u = b_2 \, \eta_1 \pm b_1 \, B, \tag{81}$$

where

$$B = \left(1 - \xi^2 - \eta_1^2\right)^{1/2}. \tag{82}$$

Now, Eq. (73) gives

$$\sin h = \frac{\xi}{\cos u}, \tag{83}$$

whereas Eqs. (74), (77)–(79), and (81) imply that

$$\cos h = \frac{\pm b_2 \, B - b_1 \, \eta_1}{\cos u}. \tag{84}$$

Finally, Eqs. (71), (78), (79), (81), and (84) can be combined to give

$$\zeta = -\omega \, f\,(2 - f) \, \cos d \, \sin d \, \eta_1 \pm \omega \, (1 - f) \, B. \tag{85}$$

The previous equation gives the z-coordinates (in the fundamental system) of the two intersection points of the straight line, normal to the fundamental plane, that passes through the point (ξ, η) in the plane. The upper sign corresponds to a point on the near side of the Earth, whereas the lower sign corresponds to a point on the far side. Obviously, it is only possible to observe a solar eclipse on the near side of the Earth. Hence, we must select the upper signs in Eqs. (81), (84), and (85). Note that these expressions are only valid if

$$\varDelta \equiv 1 - \xi^2 - \eta_1^2 \geq 0. \tag{86}$$

If $\varDelta < 0$ then this indicates that the previously mentioned straight line does not intersect the surface of the Earth. Assuming that the previous inequality is satisfied, the observer's geographic latitude and longitude can be deduced from the following equations:

$$\sin u = b_1\,B + b_2\,\eta_1, \tag{87}$$

$$\frac{\sin h}{\cos h} = \frac{\xi}{b_2\,B - b_1\,\eta_1}, \tag{88}$$

$$\tan\phi = \frac{\tan u}{1 - f}, \tag{89}$$

$$\lambda = \mu - h. \tag{90}$$

Here, use has been made of Eqs. (45), (66), (81), (83), and (84).

7 Eclipse Magnitude

The magnitude of a solar eclipse, $\mathcal{M}$, is defined as the faction of the solar diameter that is obscured. Consider the magnitude of an annular (say) eclipse for an observer, O, situated inside the penumbral cone, but outside the umbral cone, as shown in Fig. 4. In this figure, $AS\,D'A'$ represents the Sun, whose center is S, and $B'MB$ represents the Moon, whose center is M. $POUGU'T'$ is the observation plane, UU' and PP' are the projections of the Moon's umbra and penumbra on this plane, respectively, and G is the center of the Moon's shadow. V_1 is the virtual vertex of the Moon's penumbra. Finally, V_2 is the vertex of the Moon's umbra, which lies in front of the observation plane for the case of an annular eclipse. Point O has the coordinates (ξ, η, ζ), in the fundamental system, whereas point G has the coordinates (x, y, ζ). Thus, the distance $GO = m$ can be written

$$m = \left[(\xi - x)^2 + (\eta - y)^2\right]^{1/2}. \tag{91}$$

According to Sect. 3, $GU = GU' = L_2$ and $GP = GP' = L_1$. Consider the similar triangles $AD'B'$ and $U'OB'$. We can write

$$\frac{AD'}{U'O} = \frac{MS}{MG}. \tag{92}$$

Consider the similar triangles $D'A'B'$ and OPB'. We can write

$$\frac{D'A'}{OP} = \frac{MS}{MG}. \tag{93}$$

It follows that

$$\mathcal{M} = \frac{D'A'}{AD' + D'A'} = \frac{OP}{U'O + OP} = \frac{GP - GO}{GU' + GO + (GP - GO)} = \frac{GP - GO}{GU' + GP}. \tag{94}$$

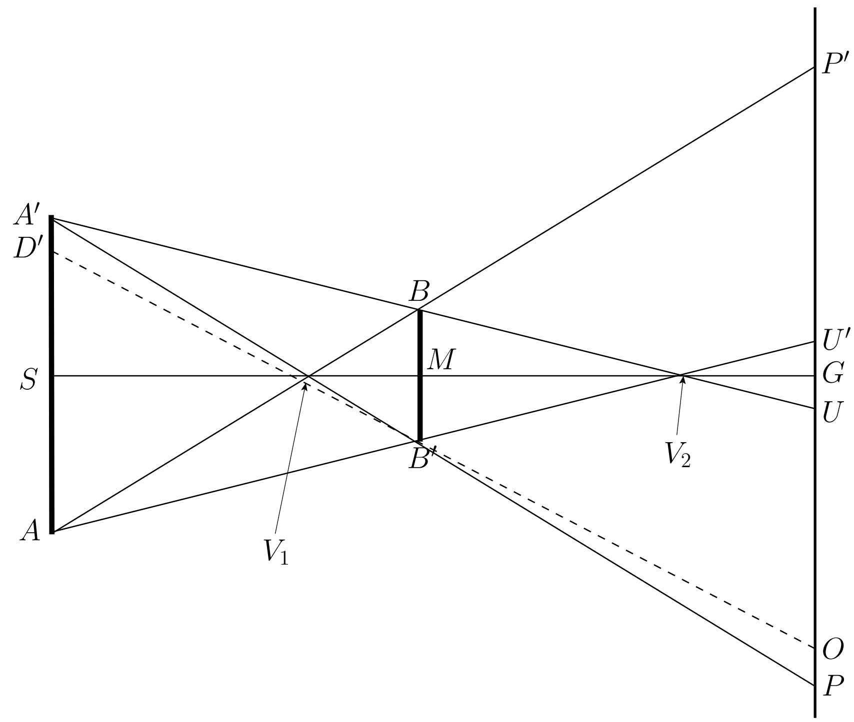

Figure 4: Eclipse magnitude.

Hence,

$$\mathcal{M} = \frac{L_1 - m}{L_1 + L_2}. \tag{95}$$

If the observer were situated on the edge of the umbral cone at U or U' then $m = L_2$, and the previous formula yields

$$\mathcal{M} = \frac{L_1 - L_2}{L_1 + L_2}. \tag{96}$$

In fact, this equation applies throughout the annular phase of the eclipse.

If the central phase is total, rather than annular, then $m = -L_2$ when the observer lies on the edge of the umbral cone. Consequently, the eclipse magnitude is unity, and remains so throughout the total phase. However, the previous formula can also be applied to total eclipses. In this case, it yields the factor by which the apparent size of the Moon exceeds that of the Sun, as seen by an observer on the eclipse's central line.

8 Eclipse Duration

Consider a point of given geographic latitude and longitude on the Earth's surface. Suppose that this point coincides with the center of the Moon's shadow at $t = 0$. It follows that $x = \xi$ and $y = \eta$ at $t = 0$. Let us calculate the duration of the total or annular phase of the eclipse seen at this point. By definition, when the total or annular phase of the eclipse is just beginning or ending, the point in question lies at the edge of the umbral cone. Now, the umbral cone intersects the observation plane, $z = \zeta$, in a circle of radius $L_2 = l_2 - \zeta\,\tan f_2$, centered on point (x, y). Moreover, the coordinates of the observation point in this plane are (ξ, η). Thus, at the beginning or end of the total or annular phase,

$$(x-\xi)^2 + (y-\eta)^2 = (l_2 - \zeta\,\tan f_2)^2. \tag{97}$$

According to Eqs. (69)–(71), and (90), at constant u and λ,

$$\xi' = \mu'\,(\zeta\,\cos d - \eta\,\sin d), \tag{98}$$

$$\eta' = \mu'\,\xi\,\sin d - d'\,\zeta, \tag{99}$$

$$\zeta' = -\mu'\,\xi\,\cos d + d'\,\eta, \tag{100}$$

where $'$ denotes a time derivative. It follows that, to first order in t,

$$x - \xi = a\,t, \tag{101}$$

$$y - \eta = b\,t, \tag{102}$$

where

$$a = \left[x' + \mu'\,(y\,\sin d - \zeta\,\cos d)\right]_{t=0}, \tag{103}$$

$$b = (y' - \mu'\,x\,\sin d + d'\,\zeta)_{t=0}. \tag{104}$$

Hence, Eq. (97) reduces to

$$\left(a^2 + b^2\right)t^2 = (L_2 + c\,t)^2, \tag{105}$$

where

$$L_2 = (l_2 - \zeta\,\tan f_2)_{t=0}, \tag{106}$$

and

$$c = l_2' + (\mu'\,x\,\cos d - d'\,y)_{t=0}\tan f_2. \tag{107}$$

Here, we have neglected the relatively weak time variation in f_2 during the total or annular phase. It follows that

$$\left(n^2 - c^2\right)t^2 - 2\,L_2\,c\,t - L_2^2 = 0, \tag{108}$$

where

$$n = \left(a^2 + b^2\right)^{1/2}. \tag{109}$$

Thus, the times at which the total or annular phase begin and end are

$$t = \pm\left(\frac{L_2}{n \pm c}\right). \tag{110}$$

The duration of the phase in question is thus

$$\tau = \frac{2\,|L_2|\,n}{n^2 - c^2}. \tag{111}$$

If all time derivatives are expressed in units of inverse days then Eq. (111) yields

$$\tau(\text{second}) = \frac{172800\,|L_2|\,n}{n^2 - c^2}. \tag{112}$$

Note that the eclipse is total if $L_2 < 0$, and annular if $L_2 > 0$.

9 Contact Times

Consider an observer situated at the edge of the penumbral cone. Let $(\xi,\,\eta)$ be the observer's fundamental coordinates in the observation plane, $z = \zeta$. The fundamental coordinates of the center of the Moon's shadow in the observation plane are $(x,\,y)$. It follows that

$$\left[(\xi - x)^2 + (\eta - y)^2\right]^{1/2} = L_1. \tag{113}$$

Hence, we can write

$$\xi = x + L_1\;\cos\alpha, \tag{114}$$

$$\eta = y + L_1\;\sin\alpha. \tag{115}$$

Consider the quantity [see Eq. (86)]

$$\Delta_1(\alpha, L_1) \equiv 1 - \xi^2 - \omega^2\,\eta^2 = 1 - (x + L_1\;\cos\alpha)^2 - \omega^2\,(y + L_1\;\sin\alpha)^2. \tag{116}$$

This quantity attains its maximal value when

$$(x + L_1\;\cos\alpha)\;\sin\alpha = \omega^2\,(y + L_1\;\sin\alpha)\;\cos\alpha. \tag{117}$$

Let

$$r = \left(x^2 + y^2\right)^{1/2}, \tag{118}$$

$$\frac{\sin\theta}{\cos\theta} = \frac{y}{x}. \tag{119}$$

It follows that $\Delta_1(\alpha, L_1)$ is maximized when

$$\alpha = \alpha_{\rm max} = \theta + \pi - \sin^{-1}(\Omega_1), \tag{120}$$

and

$$\Omega_1 = \left(\omega^2 - 1\right)\left(\sin\theta + \frac{L_1}{r}\;\sin\alpha\right)\cos\alpha. \tag{121}$$

Likewise, $\Delta_1(\alpha, L_1)$ is minimized when

$$\alpha = \alpha_{\rm min} = \theta + \sin^{-1}(\Omega_1). \tag{122}$$

The previous three equations can be conveniently solved by iteration.

At a given time, the points on the Earth's surface at which $\Delta_1(\alpha, L_1) = 0$ correspond to those points at which the partial phase of the eclipse is either beginning or ending, either at sunrise or sunset. The instants in time at which $\Delta_1(\alpha_{\rm max}, L_1) = 0$ corresponds to the first and last external contacts of the penumbral cone with the Earth's surface. These times are denoted P1 and P4, respectively. The instants in time at which $\Delta_1(\alpha_{\rm min}, L_1) = 0$ corresponds to the first and last internal contacts of the penumbral cone with the Earth's surface. These times are denoted P2 and P3, respectively. Points on the Earth's surface at which the partial phase of the eclipse is either beginning or ending at sunset only exist between times P1 and P2. Likewise, points on the Earth's surface at which the partial phase of the eclipse is either beginning or ending at sunrise only exist between times P3 and P4.

Consider an observer situated at the edge of the umbral cone. By analogy with the previous analysis, we can write

$$\xi = x + |L_2|\,\cos\alpha, \tag{123}$$

$$\eta = y + |L_2|\,\sin\alpha. \tag{124}$$

Consider the quantity [see Eq. (86)]

$$\varDelta_2(\alpha, L_2) \equiv 1 - \xi^2 - \omega^2\,\eta^2 = 1 - (x + |L_2|\,\cos\alpha)^2 - \omega^2\,(y + |L_2|\,\sin\alpha)^2. \tag{125}$$

This quantity is maximized when

$$\alpha = \alpha_{\rm max} = \theta + \pi - \sin^{-1}(\varOmega_2), \tag{126}$$

and

$$\varOmega_2 = \left(\omega^2 - 1\right)\left(\sin\theta + \frac{|L_2|}{r}\,\sin\alpha\right)\cos\alpha. \tag{127}$$

Likewise, $\varDelta_2(\alpha, L_2)$ is minimized when

$$\alpha = \alpha_{\rm min} = \theta + \sin^{-1}(\varOmega_2). \tag{128}$$

Here, r and θ are defined in Eqs. (118) and (119), respectively. The previous three equations can be conveniently solved by iteration.

At a given time, the points on the Earth's surface at which $\varDelta_2(\alpha, L_2) = 0$ correspond to those points at which the total/annular phase of the eclipse is either beginning or ending, either at sunrise or sunset. The instants in time at which $\varDelta_2(\alpha_{\rm max}, L_2) = 0$ corresponds to the first and last external contacts of the umbral cone with the Earth's surface. These times are denoted U1 and U4, respectively. The instants in time at which $\varDelta_2(\alpha_{\rm min}, L_2) = 0$ corresponds to the first and last internal contacts of the umbral cone with the Earth's surface. These times are denoted U2 and U3, respectively. Points on the Earth's surface at which the total/annular phase of the eclipse is either beginning or ending at sunset only exist between times U1 and U2. Likewise, points on the Earth's surface at which the total/annular phase of the eclipse is either beginning or ending at sunrise only exist between times U3 and U4.

10 Maximum Eclipse

According to Sect. 7, the eclipse magnitude can be written

$$\mathcal{M} = \frac{L_1 - m}{L_1 + L_2}, \tag{129}$$

where

$$m = \left[(\xi - x)^2 + (\eta - y)^2\right]^{1/2}. \tag{130}$$

Let (ξ, η, ζ) be the coordinates of all points on the Earth's surface at which the eclipse magnitude takes the value $\mathcal{M}$. It follows that

$$\xi = x + m\,\cos\alpha, \tag{131}$$

$$\eta = y + m\,\sin\alpha, \tag{132}$$

where

$$m = L_1 - (L_1 + L_2)\,\mathcal{M}. \tag{133}$$

At fixed location, the time of maximum eclipse magnitude corresponds to

$$\mathcal{M}' = 0, \tag{134}$$

where $'$ denotes a time derivative. It follows from Eq. (129) that $\mathcal{M}' = 0$ when

$$m' = \left(\frac{L_1'\,L_2 - L_1\,L_2'}{L_1 + L_2}\right) + m\left(\frac{L_1' + L_2'}{L_1 + L_2}\right). \tag{135}$$

Here, making use of Eqs. (42) and (44),

$$L_1' = l_1' - \zeta'\,\tan f_1, \tag{136}$$

$$L_2' = l_2' - \zeta'\,\tan f_2, \tag{137}$$

where we have neglected any time variation of the quantities f_1 and f_2.

Let $u = \xi - x$, $u' = \xi' - x'$, $v = \eta - y$, and $v' = \eta' - y'$. It follows from Eq. (130) that

$$m'\,m = u\,u' + v\,v'. \tag{138}$$

Let

$$n = (u'^2 + v'^2)^{1/2}, \tag{139}$$

$$\frac{\sin\theta}{\cos\theta} = -\frac{u'}{v'}. \tag{140}$$

It follows that $u = m\,\cos\alpha$, $v = m\,\sin\alpha$, $u' = -n\,\sin\theta$, and $v' = n\,\cos\theta$. Hence, Eq. (135) yields

$$\sin(\alpha - \theta) = \frac{1}{n}\left(\frac{L_1'\,L_2 - L_1\,L_2'}{L_1 + L_2}\right) + \frac{m}{n}\left(\frac{L_1' + L_2'}{L_1 + L_2}\right). \tag{141}$$

Hence, the eclipse magnitude is maximized when $\alpha = \alpha_\pm$, where

$$\alpha_+ = \theta + \sin^{-1}(\Lambda), \tag{142}$$

$$\alpha_- = \theta + \pi - \sin^{-1}(\Lambda), \tag{143}$$

and

$$\Lambda = \frac{1}{n}\left(\frac{L_1'\,L_2 - L_1\,L_2'}{L_1 + L_2}\right) + \frac{m}{n}\left(\frac{L_1' + L_2'}{L_1 + L_2}\right). \tag{144}$$