03 - Spin Angular Momentum

- <u>Aim of Section</u>:
  - Brief review of material on spin angular momentum presented in previous course (PHY 373).

Spin of Electron

- An electron is a point particle with a number of well-known properties:
  - Mass: $m_e = 9.10938356 \times 10^{-31}\,\mathrm{kg}$.
  - Charge: $e = -1.60217662\,10^{-19}\,\mathrm{C}$.
- However, an electron also possesses an intrinsic (i.e., non-orbital) angular momentum known as spin. (If we imagine that an electron is a small sphere then spin would be angular momentum associated with sphere spinning about an axis passing through its midpoint. However, electrons are actually point particles, so this analogy cannot be taken literally.)
- Magnitude of electron spin angular momentum is $\hbar/2$

Spin Operators - I

- Let operator **S** represent spin angular momentum.
- Assume that Cartesian components of **S** satisfy analogous commutation relations to components of orbital angular momentum (these commutation relations are characteristic of any type of angular momentum):

$$[S_x, S_y] = \mathrm{i}\,\hbar\, S_z, \tag{1}$$

$$[S_y, S_z] = \mathrm{i}\,\hbar\, S_x, \tag{2}$$

$$[S_z, S_x] = \mathrm{i}\,\hbar\, S_y. \tag{3}$$

- Magnitude-squared of spin angular momentum represented by

$$S^2 = S_x^2 + S_y^2 + S_z^2.$$

- Follows that

$$[S^2, S_x] = [S^2, S_y] = [S^2, S_z] = 0.$$

Spin Operators - II

- Previous commutation relations imply that, at most, can simultaneously measure S^2 and one Cartesian component of $\mathbf{S}$. Choose to measure S^2 and S_z.
- Can also define
$$S_\pm = S_x \pm \mathrm{i}\, S_y.$$
- Easily demonstrated that
$$S_+\, S_- = S^2 - S_z^2 + \hbar\, S_z, \tag{4}$$
$$S_-\, S_+ = S^2 - S_z^2 - \hbar\, S_z, \tag{5}$$
$$[S_+, S_z] = -\hbar\, S_+, \tag{6}$$
$$[S_-, S_z] = +\hbar\, S_-. \tag{7}$$

Spin Space - I

- Unlike regular wavefunctions, spin wavefunctions do not exist in real space. Likewise, spin operators cannot be expressed as differential operators in real space.
- Spin wavefunctions exist in abstract (complex) vector space. Different members of this space correspond to different spin configurations of electron.
- Only directions of vectors have any physical significance (just as only shapes of regular wavefunctions have physical significance).
- Thus, if vector χ corresponds to a particular spin state then $c\,\chi$ corresponds to same state, where c is a complex number.

Spin Space - II

- Expect spin states of electron to be superposable, because superposability of states is one of fundamental assumptions of quantum mechanics.
- Follows that vectors making up our vector space must also be superposable. Thus, if χ_1 and χ_2 are two vectors corresponding to two different spin states then $c_1\,\chi_1 + c_2\,\chi_2$ is another vector corresponding to state obtained by superposing c_1 times first state with c_2 times second state (where c_1 and c_2 are complex numbers).
- Finally, dimensionality of our vector space is simply number of linearly independent vectors required to span it (i.e., number of linearly independent spin states).

Spin Space - III

- Can define lengths of our vectors by introducing a second, or dual, vector space whose elements are in one to one correspondence with elements of our first space.
- Let element of second space that corresponds to element χ of first space be called $\chi^{\dagger}$.
- Element of second space that corresponds to $c\,\chi$ is $c^{*}\,\chi^{\dagger}$.
- Assume that it is possible to combine χ and $\chi^{\dagger}$ in multiplicative fashion to generate a real positive-definite number that we interpret as length, or norm, of χ.
- Let us denote this number $\chi^{\dagger}\chi$. Thus, we have

$$\chi^{\dagger}\,\chi \geq 0$$

for all χ.

Spin Space - IV

- Also assume that it is possible to combine unlike states in an analogous multiplicative fashion to produce complex numbers.
- Product of two unlike states, χ and χ', is denoted $\chi^\dagger\,\chi'$.
- Two states, χ and χ', are said to be orthogonal, or independent, if $\chi^\dagger\,\chi' = 0$.

Spin Space - V

- When a general spin operator, A, operates on a general spin-state, χ, it converts it into a different spin-state which we shall denote $A\,\chi$.
- Dual of this state is $(A\,\chi)^\dagger \equiv \chi^\dagger\,A^\dagger$, where $A^\dagger$ is Hermitian conjugate of A.
- An eigenstate of A corresponding to eigenvalue a satisfies

$$A\,\chi_a = a\,\chi_a.$$

- If A corresponds to physical variable then measurement of A will result in one of its eigenvalues.
- In order to ensure that these eigenvalues are all real, A must be Hermitian: i.e., $A^\dagger = A$.

Spin Space - VI

- Expect the χ_a to be mutually orthogonal.
- We can also normalize them such that they all have unit length. In other words,

$$\chi_a^\dagger \chi_{a'} = \delta_{a,a'}.$$

- General normalized spin state can be written as a superposition of normalized eigenstates of A: i.e.,

$$\chi = \sum_a c_a \chi_a.$$

- Measurement of A will then yield result a with probability $|c_a|^2$. Immediately after measurement, system left in state χ_a.

Spin Eigenstates - I

- Let simultaneous eigenstates of S^2 and S_z take form

$$S^2\,\chi_{s,m_s} = s\,(s+1)\,\hbar^2\,\chi_{s,m_s},$$
$$S_z\,\chi_{s,m_s} = m_s\,\hbar\,\chi_{s,m_s},$$

  where s and m_s are real numbers.
- Easily demonstrated from commutation relations (6) and (7) that

$$S_z\,(S_+\,\chi_{s,m_s}) = (m_s+1)\,\hbar\,(S_+\,\chi_{s,m_s}),$$
$$S_z\,(S_-\,\chi_{s,m_s}) = (m_s-1)\,\hbar\,(S_-\,\chi_{s,m_s}).$$

- Clear that S_+ and S_- are raising and lowering operators, respectively, for m_s. Note that $S_+^\dagger = S_-$.

Spin Eigenstates - II

- Eigenstates of S_z and S^2 assumed to be orthonormal: i.e.,

$$\chi^\dagger_{s,m_s}\,\chi_{s',m_s'} = \delta_{s,s'}\,\delta_{m_s,m_s'}.$$

- Consider wavefunction $\chi = S_+\,\chi_{s,m_s}$. Because we know that $\chi^\dagger\,\chi \geq 0$, it follows that

$$\begin{aligned}(S_+\,\chi_{s,m_s})^\dagger\,(S_+\,\chi_{s,m_s}) &= \chi^\dagger_{s,m_s}\,S_+^\dagger\,S_+\,\chi_{s,m_s}\\ &= \chi^\dagger_{s,m_s}\,S_-\,S_+\,\chi_{s,m_s} \geq 0.\end{aligned}$$

- (5) yields

$$s\,(s+1) \geq m_s\,(m_s+1).$$

- Similarly, if $\chi = S_-\,\chi_{s,m_s}$ then we obtain

$$s\,(s+1) \geq m_s\,(m_s-1).$$

Spin Eigenstates - III

- Assuming that $s \geq 0$, previous two inequalities imply that

$$-s \leq m_s \leq s.$$

- Clearly, at fixed s, there is both a maximum and a minimum possible value that m_s can take.
- Let $m_{s\,min}$ be minimum possible value of m_s. Follows that

$$S_-\,\chi_{s,m_{s\,min}} = 0.$$

  Otherwise, would exist quantum state with lower value of m_s.
- Follows from (4) that

$$S^{\,2} = S_+S_- + S_z^{\,2} - \hbar\,S_z.$$

Spin Eigenstates - IV

- Hence,

$$S^2\,\chi_{s,m_{s\,min}} = (S_+\,S_- + S_z^2 - \hbar\,S_z)\,\chi_{s,m_{s\,min}},$$

giving

$$s\,(s+1) = m_{s\,min}\,(m_{s\,min} - 1).$$

- Assuming that $m_{s\,min} < 0$, this equation yields

$$m_{s\,min} = -s.$$

- Likewise, it is easily demonstrated that

$$m_{s\,max} = +s.$$

Spin Eigenstates - V

- Acting on $\chi_{s,m_{s\,min}}$ a finite number of times with S_+ must convert it into $\chi_{s,m_{s\,max}}$. (Note that $S_+\,\chi_{s,m_{s\,max}} = 0$.) Each operation increases m_s by unity.
- Conclude that

$$m_{s\,max} - m_{s\,min} = 2\,s = k,$$

where k is non-negative integer. Hence, quantum number s can either take integer or half-integer values. (Orbital angular momentum can only take integer values, because of need to represent eigenstates as well-behaved real space wavefunctions. Spin angular momentum free to take half-integer values.)

- For electron, $s = 1/2$ and $m_s = \pm 1/2$.

Electron Spin Space - I

- Let us denote two independent spin eigenstates of an electron as

$$\chi_\pm \equiv \chi_{1/2,\pm 1/2}.$$

- Follows that

$$S^2\,\chi_\pm = \frac{3}{4}\,\hbar^2\,\chi_\pm,$$

$$S_z\,\chi_\pm = \pm\frac{1}{2}\,\hbar\,\chi_\pm.$$

- Note that χ_+ corresponds to an electron whose spin angular momentum vector has a positive component along z-axis. Loosely speaking, we could say that spin vector points in $+z$-direction (or its spin is "up").
- Likewise, χ_- corresponds to an electron whose spin points in $-z$-direction (or whose spin is "down").

Electron Spin Space - II

- These two eigenstates satisfy orthonormality requirements

$$\chi_+^\dagger\,\chi_+ = \chi_-^\dagger\,\chi_- = 1,$$
$$\chi_+^\dagger\,\chi_- = 0.$$

- A general spin state can be represented as a linear combination of χ_+ and χ_-: i.e.,

$$\chi = c_+\,\chi_+ + c_-\,\chi_-.$$

- Evident that electron spin space is two-dimensional.

Pauli Representation - I

- Have discussed spin space in rather abstract terms. Helpful to introduce particular representation of electron spin space due to Pauli.
- General spin state represented as a complex column vector in some two-dimensional space: i.e.,

$$\chi \equiv \begin{pmatrix} c_+ \\ c_- \end{pmatrix}.$$

- Corresponding dual vector represented as a row vector: i.e.,

$$\chi^\dagger \equiv (c_+^*, c_-^*).$$

- Vector used to represent spin state known as spinor.

Pauli Representation - II

- Product $\chi^\dagger\,\chi$ obtained according to ordinary rules of matrix multiplication: i.e.,

$$\chi^\dagger\,\chi = (c_+^*, c_-^*)\begin{pmatrix} c_+ \\ c_- \end{pmatrix} = c_+^*\,c_+ + c_-^*\,c_- = |c_+|^2 + |c_-|^2 \geq 0.$$

- Product $\chi^\dagger\,\chi'$ of different spin states also obtained from rules of matrix multiplication: i.e.,

$$\chi^\dagger\,\chi' = (c_+^*, c_-^*)\begin{pmatrix} c_+' \\ c_-' \end{pmatrix} = c_+^*\,c_+' + c_-^*\,c_-'.$$

Pauli Representation - III

- General spin operator A is represented as a 2×2 matrix that operates on a spinor: i.e.,

$$A\,\chi \equiv \begin{pmatrix} A_{11}, & A_{12} \\ A_{21}, & A_{22} \end{pmatrix} \begin{pmatrix} c_+, \\ c_- \end{pmatrix}.$$

- Hermitian conjugate of A is represented by transposed complex conjugate of matrix used to represent A: i.e.,

$$A^\dagger \equiv \begin{pmatrix} A_{11}^*, & A_{21}^* \\ A_{12}^*, & A_{22}^* \end{pmatrix}.$$

Pauli Representation - IV

- Let us represent spin eigenstates χ_+ and χ_- as

$$\chi_+ \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and

$$\chi_- \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

- Convenient to write spin operators S_i (where $i = 1, 2, 3$ corresponds to x, y, z) as

$$S_i = \frac{\hbar}{2}\,\sigma_i.$$

Here, σ_i are dimensionless 2×2 matrices.

Pauli Representation - V

- According to (1)–(3), the σ_i satisfy commutation relations

$$[\sigma_x, \sigma_y] = 2\,\mathrm{i}\,\sigma_z,$$
$$[\sigma_y, \sigma_z] = 2\,\mathrm{i}\,\sigma_x,$$
$$[\sigma_z, \sigma_x] = 2\,\mathrm{i}\,\sigma_y.$$

- Furthermore,

$$\sigma_z\,\chi_\pm = \pm\chi_\pm.$$

Pauli Representation - VI

- Easily demonstrated that previous four equations satisfied by Pauli matrices:

$$\sigma_x \equiv \begin{pmatrix} 0, & 1 \\ 1, & 0 \end{pmatrix},$$

$$\sigma_y \equiv \begin{pmatrix} 0, & -\mathrm{i} \\ \mathrm{i}, & 0 \end{pmatrix},$$

$$\sigma_z \equiv \begin{pmatrix} 1, & 0 \\ 0, & -1 \end{pmatrix}.$$

- Fact that Pauli matrices are Hermitian (i.e., $\sigma_i = \sigma_i^{T\,*}$) demonstrates that S_i are Hermitian operators.

Pauli Representation - VII

- General spinor takes form

$$\chi = c_+\,\chi_+ + c_-\,\chi_- = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}.$$

- If spinor is properly normalized then

$$\chi^\dagger\,\chi = |c_+|^2 + |c_-|^2 = 1.$$

- Can interpret $|c_+|^2$ as probability that an observation of S_z will yield result $+\hbar/2$, and $|c_-|^2$ as probability that an observation of S_z will yield result $-\hbar/2$.

Magnetic Moment - I

- Just as an electron possess a magnetic moment associated with its orbital angular momentum, it also possesses a moment associated with its spin angular momentum.
- In fact,

$$\boldsymbol{\mu} = -\frac{e}{2\,m_e}\,(\mathbf{L} + g\,\mathbf{S}).$$

- Here,

$$g = 2 + \frac{\alpha}{2\pi} + \mathcal{O}(\alpha^2)$$

  is dimensionless gyromagnetic ratio.
- Furthermore,

$$\alpha = \frac{e^2}{2\,\epsilon_0\,h\,c} \simeq \frac{1}{137}$$

  is dimensionless fine-structure constant.

Magnetic Moment - II

- Fact that $g = 2$, rather than $g = 1$, is relativistic effect. Can be accounted for via relativistically invariant Dirac equation.
- Small correction to $g = 2$ comes from quantum field theory.