02 - Orbital Angular Momentum

- Aim of Section:
  - Brief review of material on orbital angular momentum presented in previous course (PHY 373).

Angular Momentum - I

- In classical mechanics, orbital angular momentum, $\mathbf{L}$, of point particle of position vector, $\mathbf{x}$, and linear momentum, $\mathbf{p}$, is

$$\mathbf{L} = \mathbf{x} \times \mathbf{p}.$$

- Follows that

$$L_x = y\,p_z - z\,p_y, \tag{1}$$

$$L_y = z\,p_x - x\,p_z, \tag{2}$$

$$L_z = x\,p_y - y\,p_x. \tag{3}$$

- In quantum mechanics, we represent three Cartesian components of $\mathbf{L}$ by quantum mechanical versions of above three expressions (in which components of $\mathbf{x}$ are represented as algebraic operators, and components of $\mathbf{p}$ are represented as differential operators).

Angular Momentum - II

- Note that there is no ambiguity in definitions (1)–(3) because all operators on right-hand sides commute.
- In classical mechanics, magnitude-squared of angular momentum vector given by

$$L^2 = L_x^2 + L_y^2 + L_z^2.$$

  Quantum mechanics uses same definition.
- Easily demonstrated that L_x, L_y, L_z, and L^2 are Hermitian operators. (Hw. 2, Q. 1.)

Commutation Relations

- Easily shown that (Hw. 2, Q. 2),

$$[L_x, L_y] = \mathrm{i}\,\hbar\, L_z,$$
$$[L_y, L_z] = \mathrm{i}\,\hbar\, L_x,$$
$$[L_z, L_x] = \mathrm{i}\,\hbar\, L_y.$$

- Furthermore (Hw. 2, Q. 3),

$$[L^2, L_x] = [L^2, L_y] = [L^2, L_z] = 0.$$

- Above commutation relations imply that, at most, we can simultaneously measure L^2 and one Cartesian component of $\mathbf{L}$. We shall choose to simultaneously measure L^2 and L_z.

Ladder Operators

- Helpful to define non-Hermitian ladder operators:

$$L_{\pm} = L_x \pm \mathrm{i}\, L_y.$$

- Easily demonstrated that (Hw. 2, Q. 4)

$$[L_+, L_-] = 2\,\hbar\, L_z,$$
$$[L_+, L_z] = -\hbar\, L_+,$$
$$[L_-, L_z] = +\hbar\, L_-.$$

Representation of Angular Momentum - I

- Define conventional spherical coordinates, r, θ, ϕ, where

$$x = r\,\sin\theta\,\cos\phi,$$
$$y = r\,\sin\theta\,\sin\phi,$$
$$z = r\,\cos\theta.$$

- Easily demonstrated that (Hw. 2, Q. 5)

$$L_x = -\mathrm{i}\,\hbar\left(-\sin\phi\,\frac{\partial}{\partial\theta} - \cos\phi\,\cot\theta\,\frac{\partial}{\partial\phi}\right), \tag{4}$$

$$L_y = -\mathrm{i}\,\hbar\left(\cos\phi\,\frac{\partial}{\partial\theta} - \sin\phi\,\cot\theta\,\frac{\partial}{\partial\phi}\right), \tag{5}$$

$$L_z = -\mathrm{i}\,\hbar\,\frac{\partial}{\partial\phi}, \tag{6}$$

$$L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial\phi^2}\right]. \tag{7}$$

Representation of Angular Momentum - II

- Follows that

$$L_{\pm} = \hbar\,\mathrm{e}^{\pm\,\mathrm{i}\,\phi}\left(\pm\frac{\partial}{\partial\theta} + \mathrm{i}\,\cot\theta\,\frac{\partial}{\partial\phi}\right). \tag{8}$$

- Note that all angular momentum operators are functions of angular coordinates, θ and ϕ, but are completely independent of radial coordinate, r

Eigenstates of Angular Momentum - I

- Search for properly normalized simultaneous eigenstates of L^2 and L_z corresponding to eigenvalues $l\,(l+1)\,\hbar^2$ and $m\,\hbar$, respectively.
- m and l are dimensionless, because $\hbar$ has dimensions of angular momentum.
- m and $l\,(l+1)$ are real, because L^2 and L_z are Hermitian operators.
- So, we have

$$L^2\,\psi_{l,m} = l\,(l+1)\,\hbar^2\,\psi_{l,m}, \tag{9}$$

$$L_z\,\psi_{l,m} = m\,\hbar\,\psi_{l,m}, \tag{10}$$

$$\int \psi^*_{l,m}\,\psi_{l,m}\,d^3\mathbf{x} = 1. \tag{11}$$

Eigenstates of Angular Momentum - II

- Try separable solution

$$\psi_{l,m}(\mathbf{x}) = R_{l,m}(r)\,\Theta_{l,m}(\theta)\,\Phi_{l,m}(\phi). \tag{12}$$

- Given that $d^3\mathbf{x} = r^2 \sin\theta\, dr\, d\theta\, d\phi$, (11) implies that

$$\int_0^\infty R^*_{l,m}(r)\,R_{l,m}(r)\,r^2\,dr = 1, \tag{13}$$

$$\int_0^\pi \Theta^*_{l,m}(\theta)\,\Theta_{l,m}(\theta)\,\sin\theta\,d\theta = 1, \tag{14}$$

$$\oint \Phi^*_{l,m}(\phi)\,\Phi_{l,m}(\phi)\,d\phi = 1. \tag{15}$$

Eigenstates of Angular Momentum - III

- (6), (10), and (12) yield

$$
\begin{aligned}
L_z\,\psi_{l,m} &= -\mathrm{i}\,\hbar\,\frac{\partial}{\partial\phi}\left[R_{l,m}(r)\,\Theta_{l,m}(\theta)\,\Phi_{l,m}(\phi)\right]\\
&= R_{l,m}(r)\,\Theta_{l,m}(\theta)\left(-\mathrm{i}\,\hbar\,\frac{d\Phi_{l,m}}{d\phi}\right)\\
&= m\,\hbar\,R_{l,m}(r)\,\Theta_{l,m}(\theta)\,\Phi_{l,m}(\phi),
\end{aligned}
$$

which implies that

$$
-\mathrm{i}\,\frac{d\Phi_{l,m}}{d\phi} = m\,\Phi_{l,m}.
$$

Eigenstates of Angular Momentum - IV

- Solution of previous equation, subject to normalization condition (15), is

$$\Phi_m(\phi) = \frac{e^{\,i\,m\,\phi}}{\sqrt{2\pi}}. \tag{16}$$

- Have, dropped l subscript because Φ_m obviously does not depends on l.
- Now, $\Phi_m(\phi)$ must be a single-valued function of ϕ, otherwise wavefunction would be multi-valued, which makes no physical sense. Hence, we deduce that m is an integer.
- Easily seen that the Φ_m satisfy orthonormality constraint

$$\oint \Phi_m^*(\phi)\,\Phi_{m'}(\phi)\,d\phi = \delta_{m,m'}. \tag{17}$$

Eigenstates of Angular Momentum - V

- According to (7), (9), (12), and (16),

$$L^2\,\psi_{l,m} = -\hbar^2\left[\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) - \frac{m^{\,2}}{\sin^2\theta}\right] R_{l,m}(r)\,\Theta_{l,m}(\theta)\,\Phi_m(\phi)$$
$$= l\,(l+1)\,\hbar^2\,R_{l,m}(r)\,\Theta_{l,m}(\theta)\,\Phi_m(\phi).$$

- Hence, we deduce that

$$\frac{1}{\sin\theta}\,\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta_{l,m}}{d\theta}\right) + \left[l\,(l+1) - \frac{m^{\,2}}{\sin^2\theta}\right]\Theta_{l,m} = 0.$$

Eigenstates of Angular Momentum - VI

- Let $\mu = \cos\theta$. Previous equation becomes

$$\frac{d}{d\mu}\left[(1-\mu^2)\,\frac{d\Theta_{l,m}}{d\mu}\right] + \left[l\,(l+1) - \frac{m^2}{1-\mu^2}\right]\Theta_{l,m} = 0.$$

- Previous equation known as associated Legendre equation. Equation singular at $\mu = \pm 1$ (i.e., $\theta = 0, \pi$) where spherical coordinate system becomes singular.
- Solutions that are well-behaved at $\mu = \pm 1$ are known as associated Legendre functions, denoted $P_l^m(\mu)$.
- Such solutions can only be found when

$$l = 0, 1, 2, 3, \cdots,$$

$$-l \leq m \leq l.$$

Eigenstates of Angular Momentum - VII

- Associated Legendre functions take form

$$P_l^m(\mu) = \frac{(-1)^{l+m}}{2^l\, l!}\,(1-\mu^2)^{m/2}\,\frac{d^{l+m}}{d\mu^{l+m}}\,(1-\mu^2)^l,$$

  for $m \geq 0$.

- Can see why m cannot exceed l. $(1-\mu^2)^l$ is polynomial of degree $2\,l$. Polynomial annihilated if differentiated w.r.t. μ more than $2\,l$ times.

- Have

$$P_l^{-m} = (-1)^m\,\frac{(l-m)!}{(l+m)!}\,P_l^m. \tag{18}$$

Eigenstates of Angular Momentum - VIII

- Associated Legendre functions satisfy

$$\int_{-1}^{1} P_l^m\,P_{l'}^m\,d\mu = \frac{2\,(l+m)!}{(2l+1)\,(l-m)!}\,\delta_{l,l'}. \tag{19}$$

- Clear that $\Theta_{l,m}(\theta) \propto P_l^m(\cos\theta)$. In fact, (14) and (19) imply that

$$\Theta_{l,m}(\theta) = \left[\frac{2l+1}{2}\,\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta), \tag{20}$$

  for $m \geq 0$.

- Follows from (18) and (20) that

$$\Theta_{l,-m} = (-1)^m\,\Theta_{l,m}. \tag{21}$$

Eigenstates of Angular Momentum - IX

- Finally, (19) and (20) imply that

$$\int_0^{\pi} \Theta^*_{l,m}(\theta)\,\Theta_{l',m}(\theta)\,\sin\theta\,d\theta = \delta_{l,l'}. \tag{22}$$

Eigenstates of Angular Momentum - X

- Conclude that simultaneous eigenstates of L^2 and L_z, corresponding to eigenvalues $l\,(l+1)\,\hbar^2$ and $m\,\hbar$, are such that l is nonnegative integer, and m is integer lying in range $-l \leq m \leq l$.
- Moreover,

$$\psi_{l,m}(\mathbf{x}) = R_{l,m}(r)\, Y_l^m(\theta, \phi),$$

where $R_{l,m}(r)$ is undetermined, and the

$$\begin{aligned} Y_l^m(\theta,\phi) &= \Theta_{l,m}(\theta)\,\Phi_m(\phi) \\ &= \left[\frac{2l+1}{4\pi}\,\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta)\,\mathrm{e}^{\,\mathrm{i}\,m\,\phi} \end{aligned} \tag{23}$$

are termed spherical harmonics.

Eigenstates of Angular Momentum - XI

- Follows from (16) and (21) that

$$Y_l^{m\,*} = (-1)^m \, Y_l^{-m}$$

- Follows from (17) and (22) that

$$\oint Y_l^{m\,*} \, Y_{l'}^{m'} \, d\Omega = \delta_{l,l'} \, \delta_{m,m'},$$

  where $d\Omega = \sin\theta \, d\theta \, d\phi$ is an element of solid angle, and integral is over all solid angle.

- Note that the $Y_l^m(\theta, \phi)$ form a complete set. In other words, any single-valued, well-behaved function of θ and ϕ can be represented as a linear superposition of the $Y_l^m(\theta, \phi)$.

Raising and Lowering Operators

- The $P_l^m(\mu)$ have the following property

$$\frac{dP_l^m}{d\mu} = -\frac{1}{\sqrt{1-\mu^2}}\,P_l^{m+1} - \frac{m\,\mu}{1-\mu^2}\,P_l^m$$
$$= \frac{(l+m)\,(l-m+1)}{\sqrt{1-\mu^2}}\,P_l^{m-1} + \frac{m\,\mu}{1-\mu^2}\,P_l^m.$$

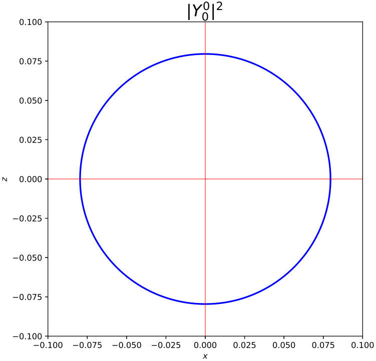
- Follows from (8) and (23) that (Hw. 2, Q. 6)

$$L_+\,Y_l^m = [l\,(l+1) - m\,(m+1)]^{1/2}\,\hbar\,Y_l^{m+1},$$
$$L_-\,Y_l^m = [l\,(l+1) - m\,(m-1)]^{1/2}\,\hbar\,Y_l^{m-1}.$$

- L_+ and L_- are termed raising and lowering operators, respectively, because they, respectively, raise and lower value of quantum number m by unity.

$l = 0$ Spherical Harmonics

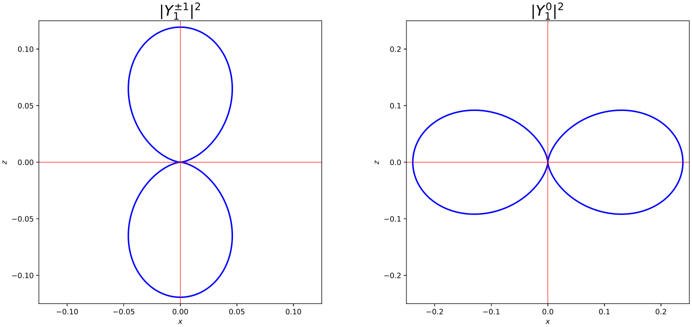
- $$Y_0^0(\theta, \phi) = \frac{1}{\sqrt{4\pi}}.$$



$l = 1$ Spherical Harmonics

- $$Y_1^{\pm 1}(\theta, \phi) = \mp \sqrt{\frac{3}{8\pi}} \sin\theta \, \mathrm{e}^{\pm \mathrm{i}\phi}.$$
- $$Y_1^0(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos\theta.$$

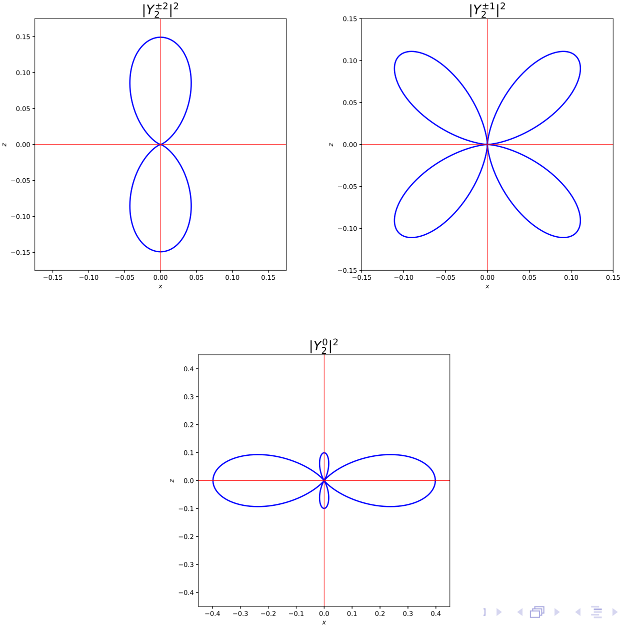
$l = 1$ Spherical Harmonics



$l = 2$ Spherical Harmonics

- $$Y_2^{\pm 2}(\theta, \phi) = \sqrt{\frac{15}{32\pi}} \sin^2\theta \, \mathrm{e}^{\pm 2\,\mathrm{i}\,\phi}.$$
- $$Y_2^{\pm 1}(\theta, \phi) = \mp\sqrt{\frac{15}{8\pi}} \sin\theta \, \cos\theta \, \mathrm{e}^{\pm\mathrm{i}\,\phi}.$$
- $$Y_2^{0}(\theta, \phi) = \sqrt{\frac{5}{16\pi}} (3 \cos^2\theta - 1).$$

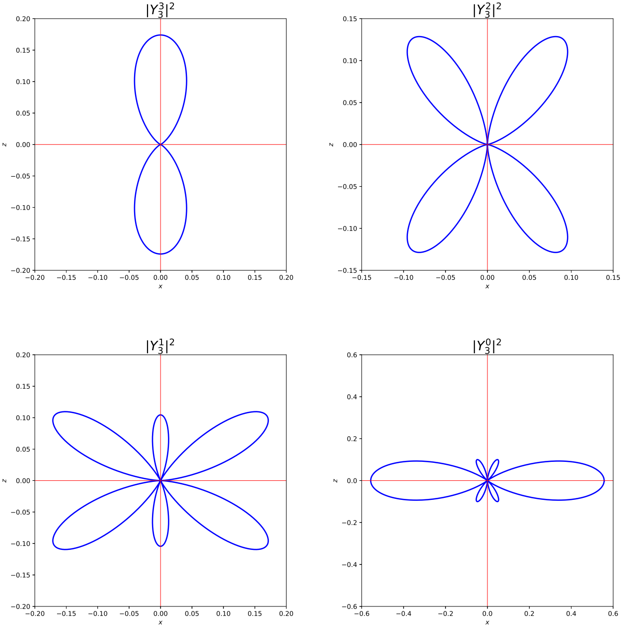
$l = 2$ Spherical Harmonics



$l = 3$ Spherical Harmonics

- $$Y_3^{\pm 3}(\theta,\phi) = \pm\sqrt{\frac{35}{64\pi}}\,\sin^3\theta\,\mathrm{e}^{\pm 3\,\mathrm{i}\,\phi}.$$
- $$Y_3^{\pm 2}(\theta,\phi) = \sqrt{\frac{105}{32\pi}}\,\sin^2\theta\,\cos\theta\,\mathrm{e}^{\pm 2\,\mathrm{i}\,\phi}.$$
- $$Y_3^{\pm 1}(\theta,\phi) = \pm\sqrt{\frac{21}{64\pi}}\,\sin\theta\,(5\,\cos^2\theta - 1)\,\mathrm{e}^{\pm\mathrm{i}\,\phi}.$$
- $$Y_3^{0}(\theta,\phi) = \sqrt{\frac{7}{16\pi}}\,\cos\theta\,(5\,\cos^2\theta - 3).$$

$l = 3$ Spherical Harmonics



Magnetic Moment

- Consider electron of charge $-e$ and mass m_e.
- Magnetic moment of electron, due to its motion in space, is

$$\boldsymbol{\mu} = -\frac{e}{2\,m_e}\,\mathbf{x}\times\mathbf{p}.$$

- Follows that

$$\boldsymbol{\mu} = -\frac{e}{2\,m_e}\,\mathbf{L}.$$

- Conclude that electron that possesses orbital angular momentum also possesses magnetic moment.
- Magnetic moments have physical consequences. Energy of magnetic moment in magnetic field $\mathbf{B}$ is $E = -\boldsymbol{\mu}\cdot\mathbf{B}$. Force acting on magnetic moment in inhomogeneous magnetic field is $\mathbf{F} = \boldsymbol{\mu}\cdot\nabla B$.