

Magnetic Island Theory - I

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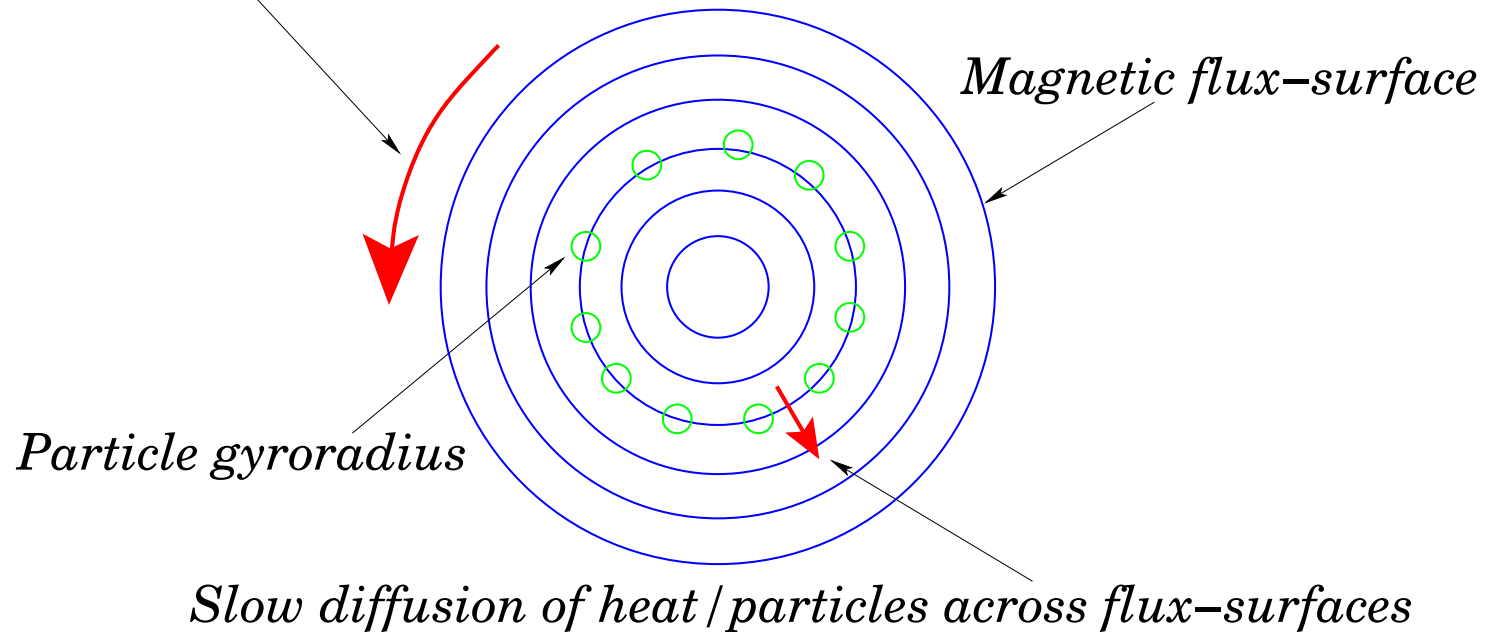
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Introduction: Toroidal Magnetic Confinement

- Toroidal magnetic confinement devices designed to trap hot plasma within set of toroidally nested magnetic flux-surfaces.
- Basic principle—charged particles free to stream along field-lines, but “stick” to magnetic flux-surfaces due to (relatively) small gyroradii.
- Energy flows rapidly along field-lines, but can only diffuse across flux-surfaces relatively slowly. Diffusion rate controlled by small-scale plasma turbulence.

Poloidal Cross-Section of Toroidal Confinement Device

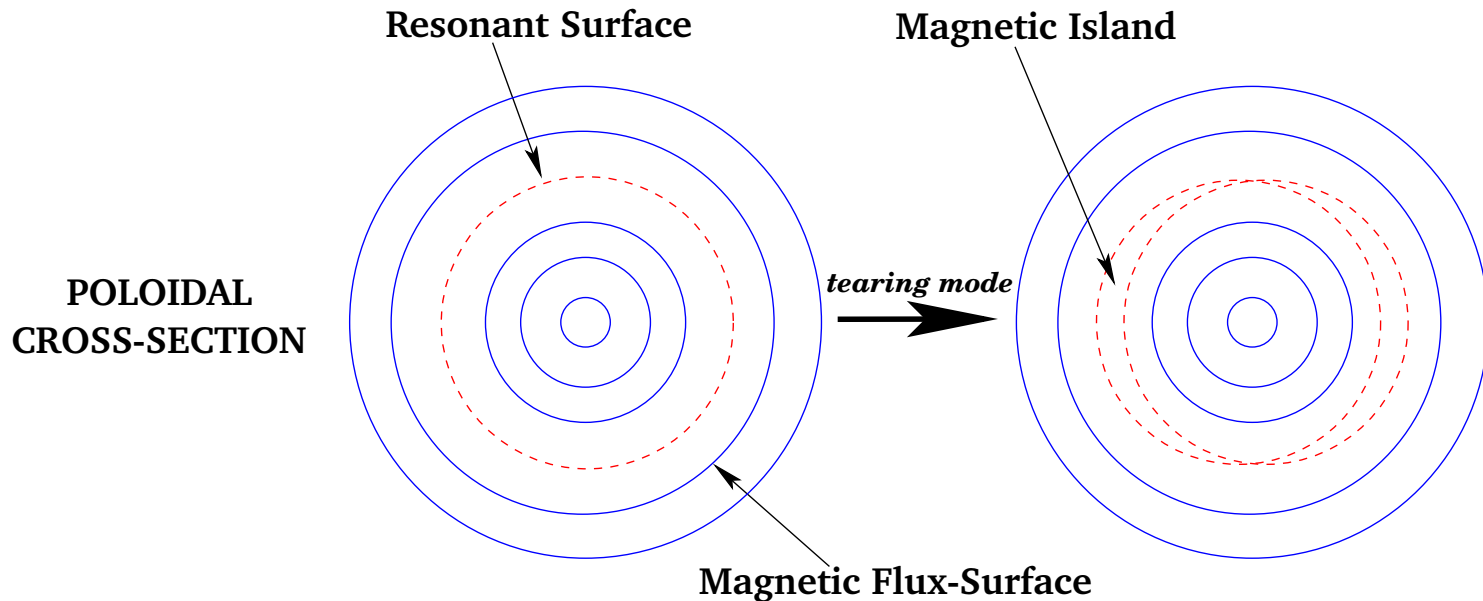
Rapid flow of heat / particles along field-lines



Introduction: Magneto-hydrodynamical Instabilities

- Two main types of MHD instability:
 - Catastrophic “ideal” instabilities, which destroy plasma in matter of micro-seconds—we know how to avoid these.
 - Slowly growing “tearing” instabilities, which reconnect magnetic flux-surfaces, thereby degrading their confinement properties—much harder to avoid.

Introduction: Tearing Modes

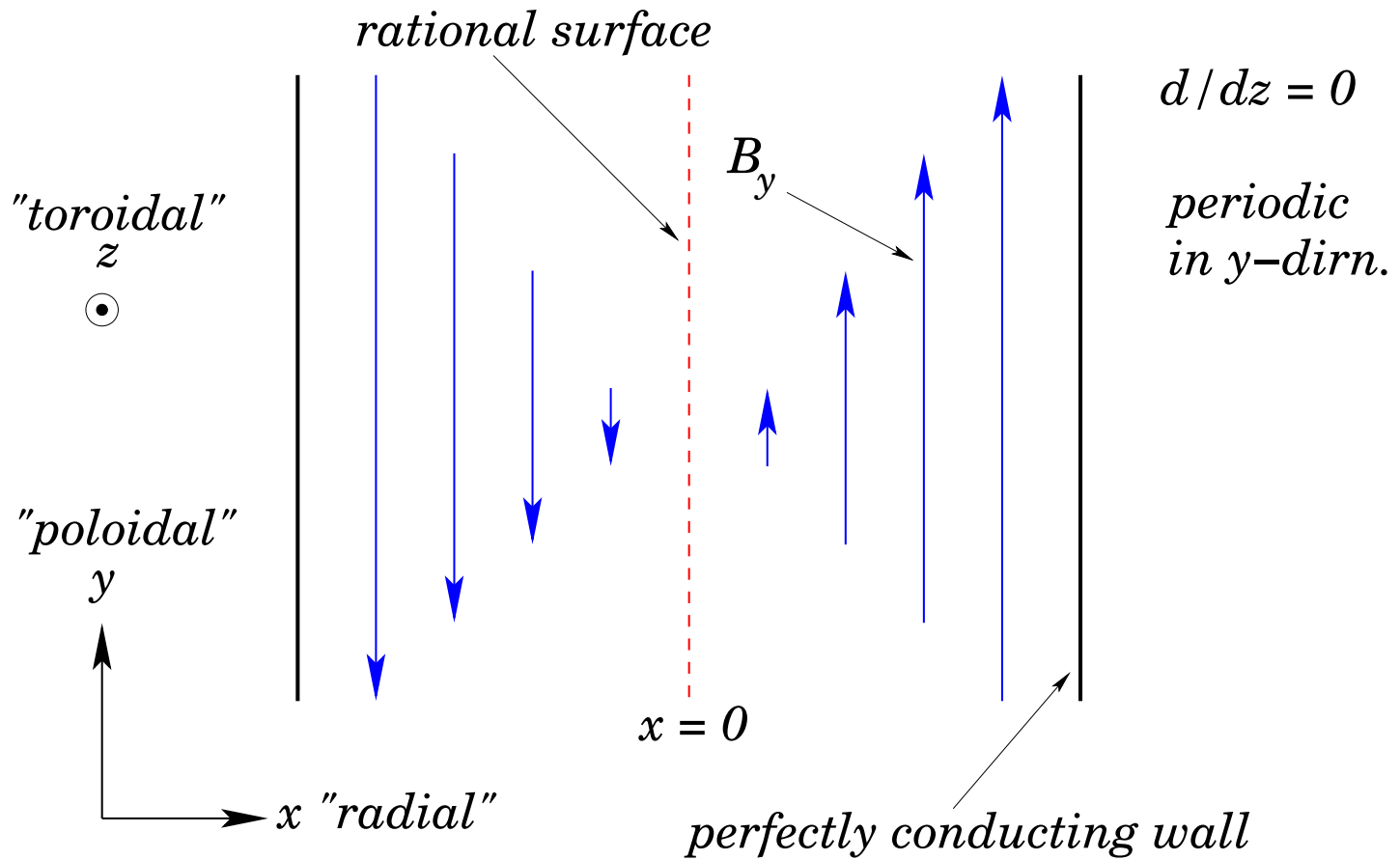


- Centered on *rational flux-surfaces* which satisfy $\vec{k} \cdot \vec{B} = 0$, where \vec{k} is wave-number, and \vec{B} is equilibrium magnetic field.
- Effectively “short-circuit” confinement by allowing heat to transit island region by rapidly flowing along field-lines, rather than slowly diffusing across flux-surfaces.

Introduction: Physics of Tearing Modes

- Tearing mode stability affected by great many factors:
 - MHD effects - current/pressure gradients across whole plasma.
 - Drift effects - diamagnetic flows, gyroviscosity, curvature drifts.
 - Neoclassical effects - poloidal flow-damping, bootstrap current.
 - Transport effects - anisotropic heat/particle transport.
 - Orbit effects - ion banana widths comparable to island width.
- This lecture will concentrate on fundamentals of MHD island theory.

Slab Approximation



MHD effects: Slab Model

- Cartesian coordinates: (x, y, z) . Let $\partial/\partial z \equiv 0$.
- All lengths/magnetic field-strengths normalized to convenient length-scale/scale field-strength. All times normalized to associated Alfvén time.
- Harris pinch equilibrium: $B_y = \tanh(x/b)$.
- Perfect wall boundary conditions at $x = \pm a$.
- Wave-number of tearing instability: $\vec{k} = (0, k, 0)$, so $\vec{k} \cdot \vec{B} = 0$ at $x = 0$. Hence, magnetic island centered at $x = 0$.

MHD effects: Model MHD equations

- Let $\vec{B} = \nabla\psi \times \vec{z}$ and $\vec{V} = \nabla\phi \times \vec{z}$, where \vec{V} is MHD fluid velocity.
- MHD equations:

$$\frac{\partial\psi}{\partial t} = [\phi, \psi] + \eta J,$$

$$\frac{\partial\mathcal{U}}{\partial t} = [\phi, \mathcal{U}] + [J, \psi] + \mu \nabla^2 \mathcal{U},$$

where $J = \nabla^2\psi$, $\mathcal{U} = \nabla^2\phi$, and $[A, B] = A_x B_y - A_y B_x$. Here, η is resistivity, and μ is viscosity. In normalized units: $\eta, \mu \ll 1$.

MHD effects: Outer Region

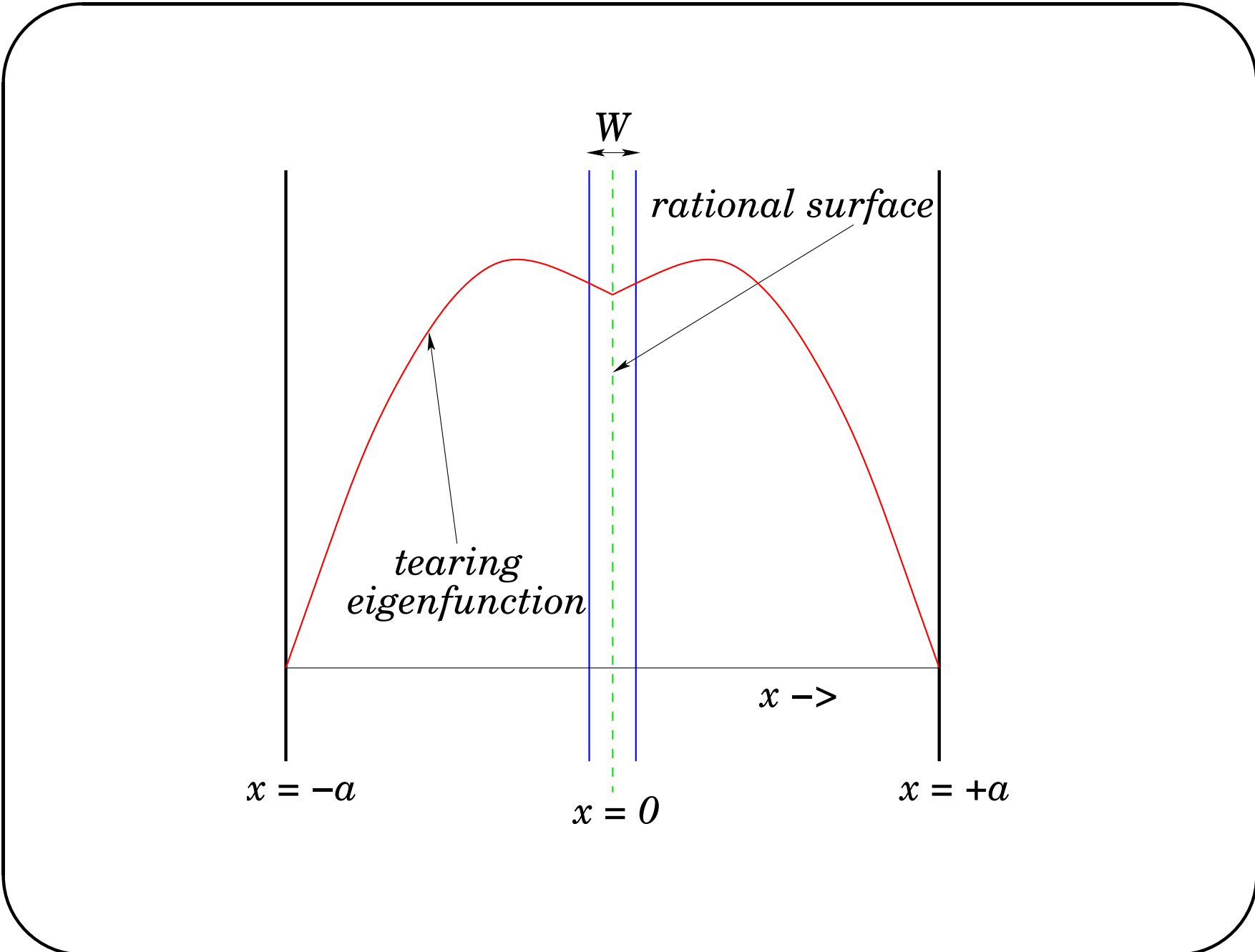
- In “outer region”, which comprises most of plasma, non-ideal effects (η and μ), and plasma inertia ($\partial/\partial t$ and $\vec{V} \cdot \nabla$), negligible.
- Vorticity equation reduces to

$$[J, \psi] \simeq 0.$$

- When linearized, obtain

$$\left(\frac{d^2}{dx^2} - k^2 \right) \psi^{(1)} - \left(\frac{d^2 B_y^{(0)} / dx^2}{B_y^{(0)}} \right) \psi^{(1)} = 0.$$

- Equation is *singular* at rational surface, $x = 0$, where $B_y^{(0)} = 0$.



MHD effects: Tearing Stability Index

- Find tearing eigenfunction, $\psi^{(1)}(x)$, which is continuous, has tearing parity [$\psi^{(1)}(-x) = \psi^{(1)}(x)$], and satisfies boundary condition $\psi^{(1)}(a) = 0$ at conducting wall.
- In general, eigenfunction has *gradient discontinuity* across rational surface (at $x = 0$). Allowed because tearing mode equation singular at rational surface.
- Tearing stability index:

$$\Delta' = \left[\frac{d \ln \psi^{(1)}}{dx} \right]_{0-}^{0+}.$$

- According to conventional MHD theory,^a tearing mode is unstable if $\Delta' > 0$.

^aH.P. Furth, J. Killeen, and M.N. Rosenbluth, Phys. Fluids **6**, 459 (1963).

MHD effects: Inner Region

- Inner region centered on $x = 0$. Of extent, $W \ll 1$, where W is magnetic island width (in x).
- In inner region, non-ideal effects, non-linear effects, and plasma inertia can all be important.
- Inner solution must be asymptotically matched to outer solution already obtained.

MHD effects: Constant- ψ Approximation

- Magnetic flux generally does not vary significantly in x over inner region:

$$|\psi(W, y) - \psi(0, y)| \ll |\psi(0, y)|.$$

- Constant- ψ approximation: treat ψ as constant in x over inner region.
- Approximation valid provided

$$|\Delta'| W \ll 1,$$

which is easily satisfied for conventional tearing modes.

MHD effects: Constant- ψ Magnetic Island

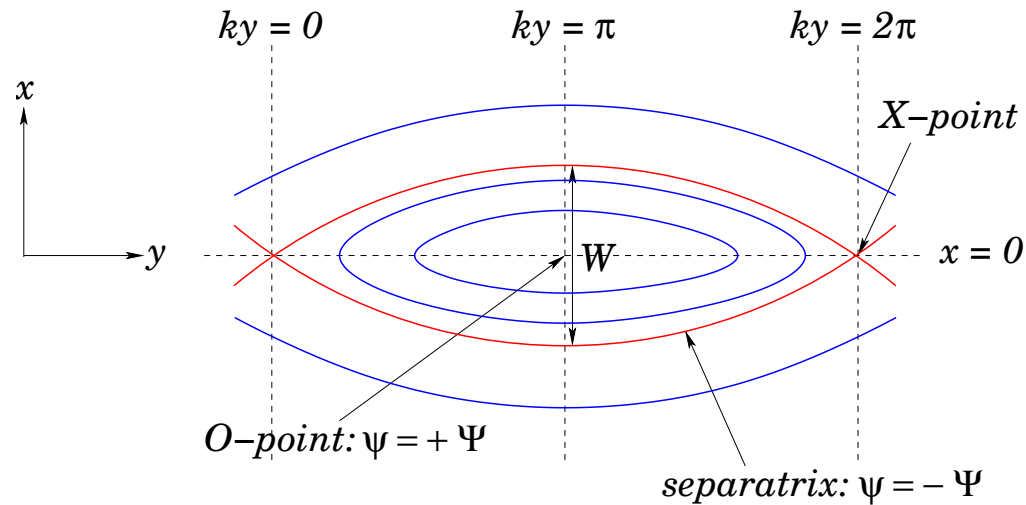
- In vicinity of rational surface:

$$\psi(x, y, t) \simeq -\frac{1}{2} x^2 + \Psi(t) \cos \theta,$$

where Ψ is “reconnected flux, and $\theta = ky$.

- Full island width

$$W = 4 \sqrt{\Psi}.$$



MHD effects: Flux-Surface Average Operator

- Flux-surface average operator is annihilator of Poisson bracket
 $[A, \psi] \equiv \vec{B} \cdot \nabla A \equiv k \times (\partial A / \partial \theta)_\psi$ for any A : *i.e.*,

$$\langle [A, \psi] \rangle = 0.$$

- Outside separatrix:

$$\langle f(\psi, \theta) \rangle = \oint \frac{f(\psi, \theta)}{|\mathbf{x}|} \frac{d\theta}{2\pi}.$$

- Inside separatrix:

$$\langle f(s, \psi, \theta) \rangle = \int_{-\theta_0}^{\theta_0} \frac{f(s, \psi, \theta) + f(-s, \psi, \theta)}{2|\mathbf{x}|} \frac{d\theta}{2\pi},$$

where $s = \text{sgn}(\mathbf{x})$, and $\mathbf{x}(s, \psi, \theta_0) = 0$.

MHD effects: MHD Flow -I

- Move to island frame. Look for steady-state solution: $\partial/\partial t = 0$.^a
- Ohm's law:

$$0 \simeq [\phi, \psi] + \eta J.$$

- Since $\eta \ll 1$, first term potentially much larger than second.
- To lowest order:

$$[\phi, \psi] \simeq 0.$$

- Follows that

$$\phi = \phi(\psi);$$

i.e., MHD flow constrained to be around flux-surfaces.

^aF.L. Waelbroeck, and R. Fitzpatrick, Phys. Rev. Lett. **78**, 1703 (1997).

MHD effects: MHD Flow - II

- Let

$$M(\psi) = \frac{d\phi}{d\psi}.$$

- Easily shown that

$$V_y = x M.$$

- $M(\psi)$ is *odd* function of x . Hence,

$$M = 0$$

inside separatrix: *i.e.*, no flow inside separatrix in island frame.
Plasma trapped within separatrix.

MHD effects: MHD Flow - III

- Vorticity equation:

$$0 \simeq -M [\mathbf{U}, \psi] + [\mathbf{J}, \psi] + \mu \nabla^4 \phi.$$

- Flux-surface average, recalling that $\langle [\mathbf{A}, \psi] \rangle = 0$:

$$\langle \nabla^4 \phi \rangle \simeq 0.$$

- Solution outside separatrix:

$$M(\psi) = \text{sgn}(x) M_0 \int_{-\Psi}^{\psi} d\psi / \langle x^4 \rangle \bigg/ \int_{-\Psi}^{-\infty} d\psi / \langle x^4 \rangle.$$

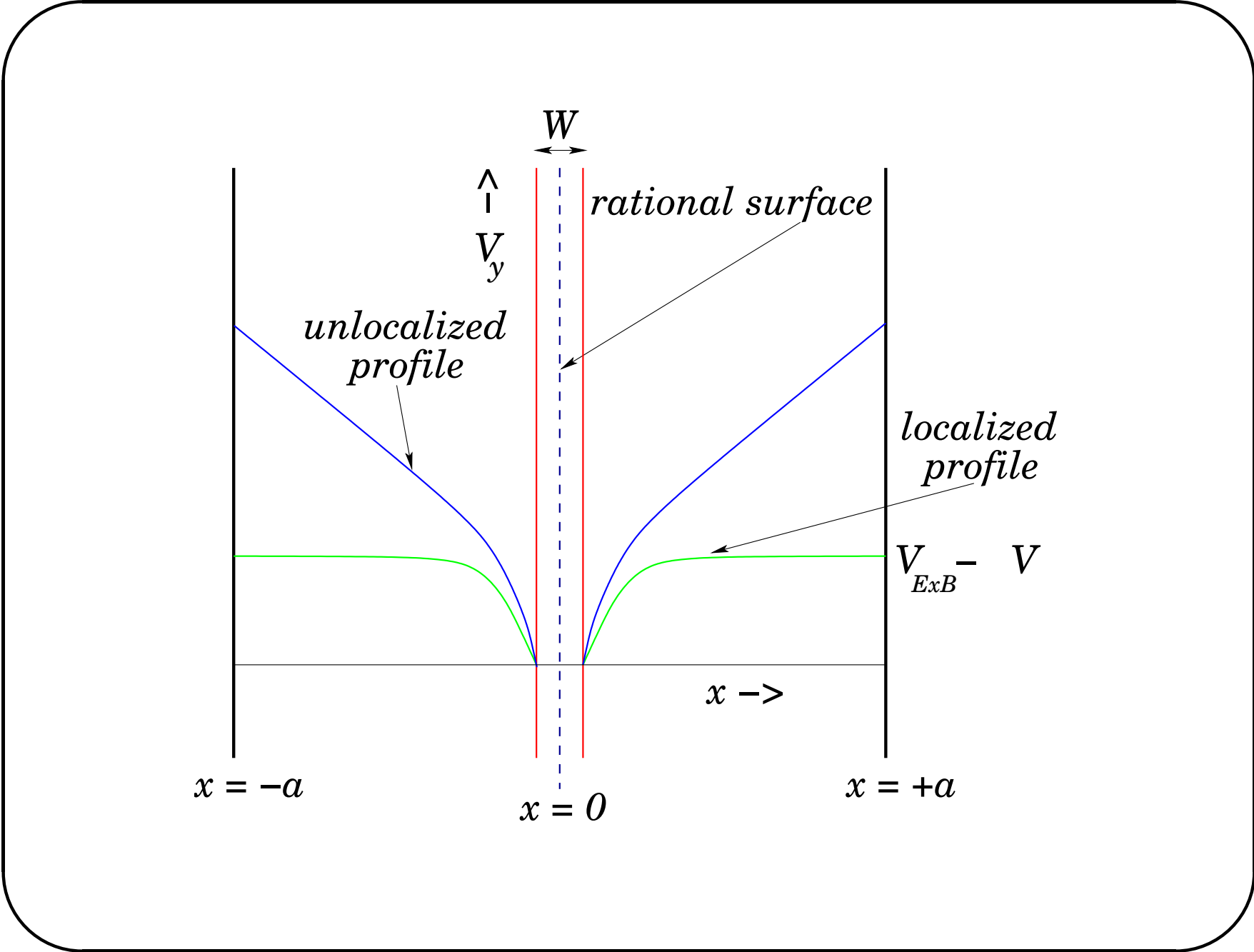
MHD effects: MHD Flow - IV

- Note

$$V_y = x M \rightarrow |x| M_0$$

as $|x|/W \rightarrow \infty$.

- V-shaped velocity profile which extends over whole plasma.
- Expect *isolated* magnetic island to have *localized* velocity profile. Suggests that $M_0 = 0$ for isolated island.
- Hence, island propagates at local $\vec{E} \times \vec{B}$ velocity.



MHD effects: MHD Flow - V

- Also expect isolated island to be subject to *zero EM torque*.
- Zero torque condition:

$$\int_{+\Psi}^{-\infty} \langle J \sin \theta \rangle d\psi \equiv \frac{1}{k\Psi} \int_{+\Psi}^{-\infty} \langle x [J, \psi] \rangle d\psi = 0.$$

- Follows from vorticity equation that

$$\lim x/w \rightarrow \infty \left[\text{sgn}(x) x^2 \frac{d}{dx} \left(\frac{1}{x} \frac{d(xM)}{dx} \right) \right] = -M_0 = 0.$$

- Hence, localized velocity profiles ($M_0 = 0$) automatically lead to zero EM torque, and *vice versa*.

MHD effects: Rutherford Equation - I

- Asymptotic matching between inner and outer regions yields:

$$\Delta' \Psi = -4 \int_{+\Psi}^{-\infty} \langle J \cos \theta \rangle d\psi.$$

- In absence of flow, in island frame, vorticity equation reduces to

$$[J, \psi] \simeq 0.$$

- Hence,

$$J = J(\psi).$$

MHD effects: Rutherford Equation - II

- Ohm's law:

$$\frac{d\Psi}{dt} \cos \theta \simeq [\phi, \psi] + \eta J(\psi).$$

- Flux surface average:

$$\frac{d\Psi}{dt} \langle \cos \theta \rangle \simeq \eta J(\psi) \langle 1 \rangle.$$

- Hence,

$$\Delta' \Psi \simeq -\frac{4}{\eta} \frac{d\Psi}{dt} \int_{+\Psi}^{-\infty} \frac{\langle \cos \theta \rangle^2}{\langle 1 \rangle} d\psi.$$

MHD effects: Rutherford Equation - III

- Evaluating flux-surface integral, obtain Rutherford equation:^a

$$\frac{0.823}{\eta} \frac{dW}{dt} \simeq \Delta'.$$

- According to Rutherford equation, island grows algebraically on resistive time-scale.
- Rutherford equation does not predict island saturation.

^aP.H. Rutherford, Phys. Fluids **16**, 1903 (1973).

MHD effects: Rutherford Equation - IV

- Higher order asymptotic matching between inner and outer regions yields:^a

$$\frac{0.823}{\eta} \frac{dW}{dt} \simeq \Delta' - 0.41 \left(-\frac{d^4 B_y^{(0)} / dx^4}{d^2 B_y^{(0)} / dx^2} \right)_{x=0} W.$$

- Hence, saturated ($d/dt = 0$) island width is

$$W_0 = \frac{\Delta'}{0.41} \left(-\frac{d^2 B_y^{(0)} / dx^2}{d^4 B_y^{(0)} / dx^4} \right)_{x=0}.$$

^aF. Militello, and F. Porcelli, Phys. Plasmas **11**, L13 (2004); D.F. Escande, and M. Ottaviani, Physics Lett. A **323**, 278 (2004).

MHD effects: Summary

- Tearing mode unstable if $\Delta' > 0$.
- Island propagates at local $\vec{E} \times \vec{B}$ velocity at rational surface.
- Island grows algebraically on resistive time-scale.
- Saturated island width:

$$W_0 = \frac{\Delta'}{0.41} \left(-\frac{d^2 B_y^{(0)} / dx^2}{d^4 B_y^{(0)} / dx^4} \right)_{x=0} .$$