Magnetic Reconnection in Tokamaks

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Lectures available online at

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Outline

1. Introduction.
2. Toroidal magnetic confinement.
3. MHD theory.
4. Profile modification.
5. Bootstrap current destabilization.
6. Drift-MHD theory.
7. Subsonic island theory.
8. Supersonic island theory.
9. Other effects.

1: Introduction

Magnetic Reconnection in Astrophysical Plasmas

- Narrow current sheets. Highly unstable to tearing instabilities.
- Very rapid (quasi-Alfvénic) reconnection rates.
- Reconnection gives rise to significant release of thermal energy, as well as copious charged particle acceleration.
- Main aims of astrophysical reconnection theory are to explain rapid onset of reconnection, to account for fast reconnection rate, and to understand energy release and particle acceleration mechanisms.

Magnetic Reconnection[a] in Tokamak Plasmas

- Extended current distributions. Comparatively stable to tearing instabilities.
- Reconnection changes topology of magnetic flux-surfaces, thereby degrading energy and particle confinement.
- Reconnection-induced energy release and charged particle acceleration completely negligible.
- Reconnection timescale irrelevant since sufficient time for tearing instability growing at smallest conceivable rate to eventually cause significant change in topology of flux-surfaces.
- Main aims of fusion reconnection theory are to understand various factors that determine stability, and final saturated amplitude, of tearing instabilities.

[a] Excluding sawtooth oscillation.

2: Toroidal Magnetic Confinement

Principles of Tokamak Confinement

- Tokamaks designed to trap hot plasma on set of axisymmetric, nested, toroidal magnetic flux-surfaces.[a]
- Fundamental principle—charged particles free to stream along magnetic field-lines, but "stick" to flux-surfaces due to their (relatively) small gyroradii.
- Heat/particles flow rapidly along field-lines, but can only diffuse relatively slowly across flux-surfaces. Diffusion rate controlled by small-scale plasma turbulence.

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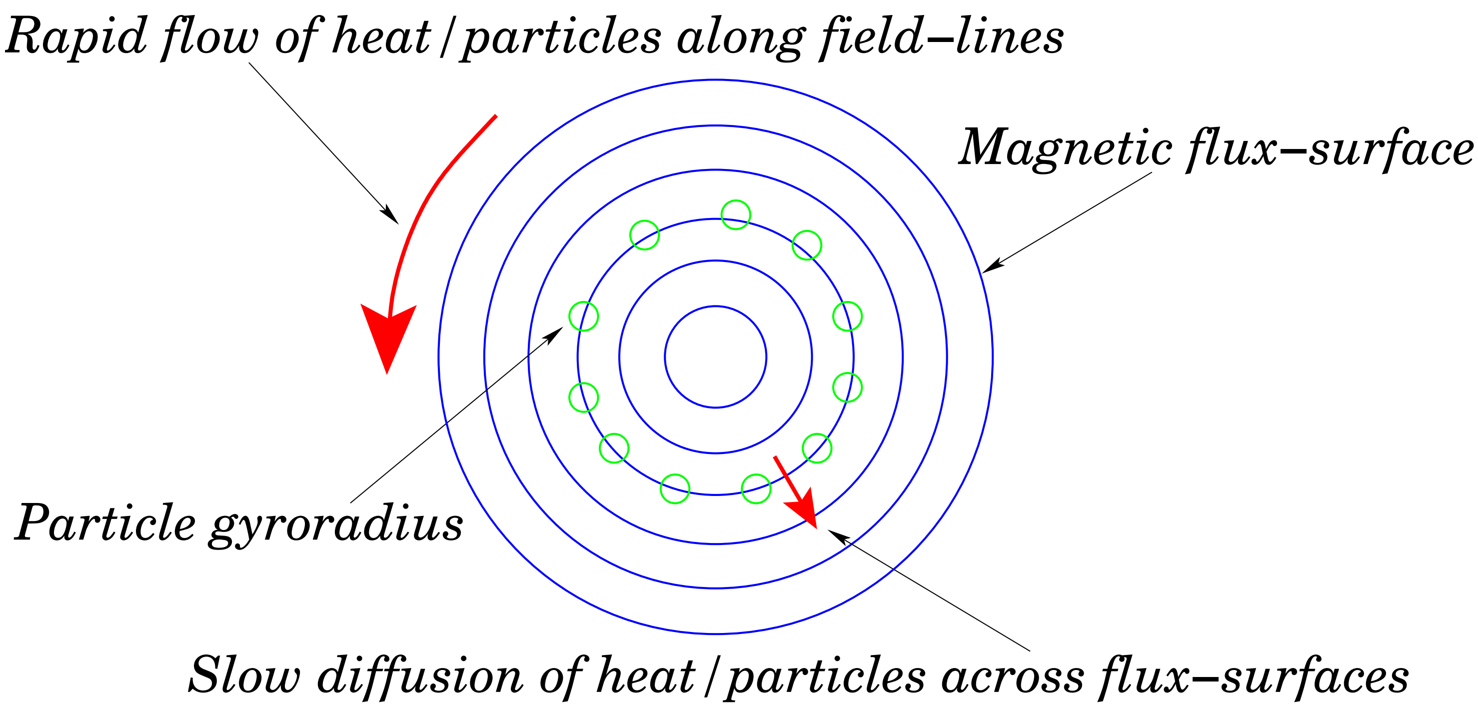
[a] *Tokamaks*, 3rd Edition, J. Wesson (Oxford University Press, 2004). *Ideal Magnetohydrodynamics*, J.P. Freidberg (Springer, 1987).

Poloidal Cross–Section of Toroidal Confinement Device

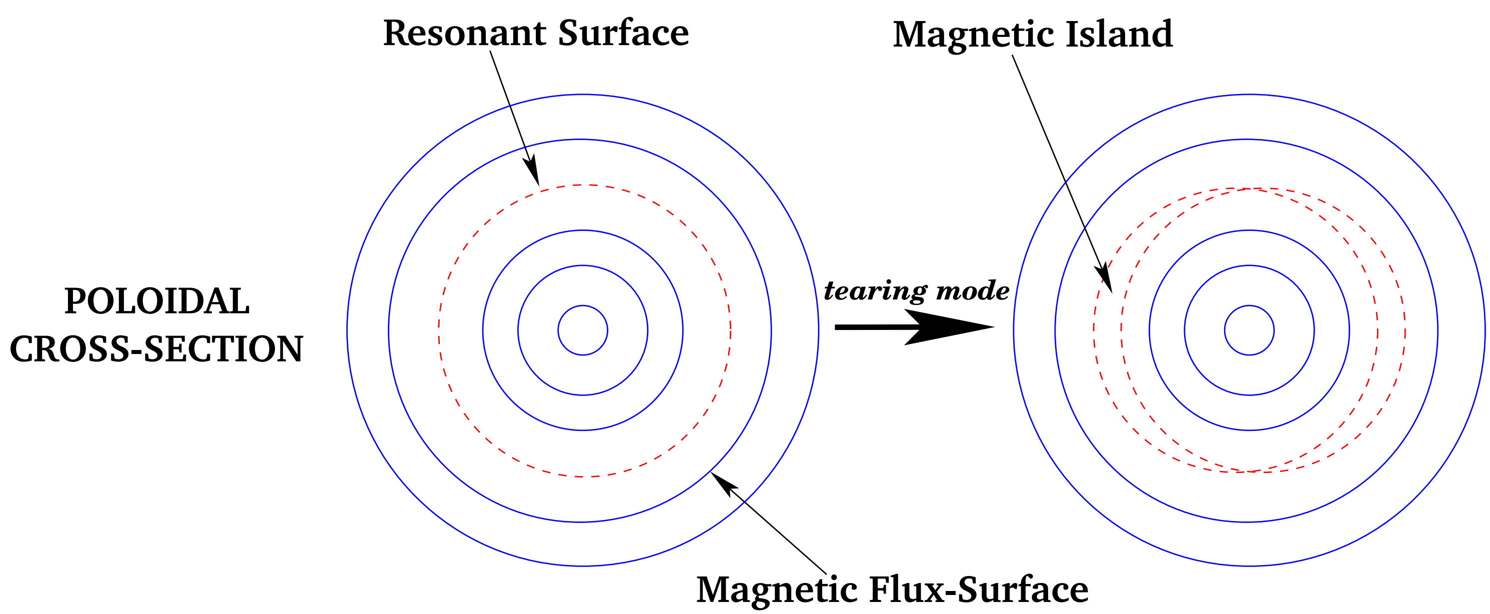


Macroscopic Instabilities

- Two main types of macroscopic instability[a] in tokamaks:
  - Catastrophic "ideal" (*i.e.*, non-reconnecting) instabilities that disrupt plasma in matter of micro-seconds—easily avoided.
  - Slowly growing "tearing" instabilities that reconnect magnetic flux-surfaces, but eventually saturate at relatively low amplitude to form *magnetic islands*, thereby degrading plasma confinement—much harder to avoid.

[a] *MHD Instabilities*, G. Bateman (MIT, 1978).

Magnetic Islands



- Centered on *rational flux-surfaces* which satisfy $\vec{k} \cdot \vec{B} = 0$, where $\vec{k}$ is wave-number of mode, and $\vec{B}$ is equilibrium magnetic field.
- Effectively "short-circuit" confinement by allowing heat/particles to transit island region by rapidly flowing along field-lines, rather than slowly diffusing across flux-surfaces.

Need for Magnetic Island Theory

- Magnetic island formation associated with *nonlinear* phase of tearing mode growth (*i.e.*, when island width becomes greater than linear layer width at rational surface).
- In very hot plasmas found in modern-day tokamaks, linear layers so thin that tearing mode already in nonlinear regime when first detected.
- Linear tearing mode theory largely irrelevant. Require nonlinear magnetic island theory to explain experimental observations.

3: MHD Theory

Introduction

- Tearing modes are macroscopic instabilities that affect whole plasma. Natural to investigate them using some form of *fluid-theory*.
- Simplest fluid theory is well-known *magnetohydrodynamical approximation*,[a] which effectively treats plasma as *single-fluid*.
- Use *slab approximation* to simplify analysis.

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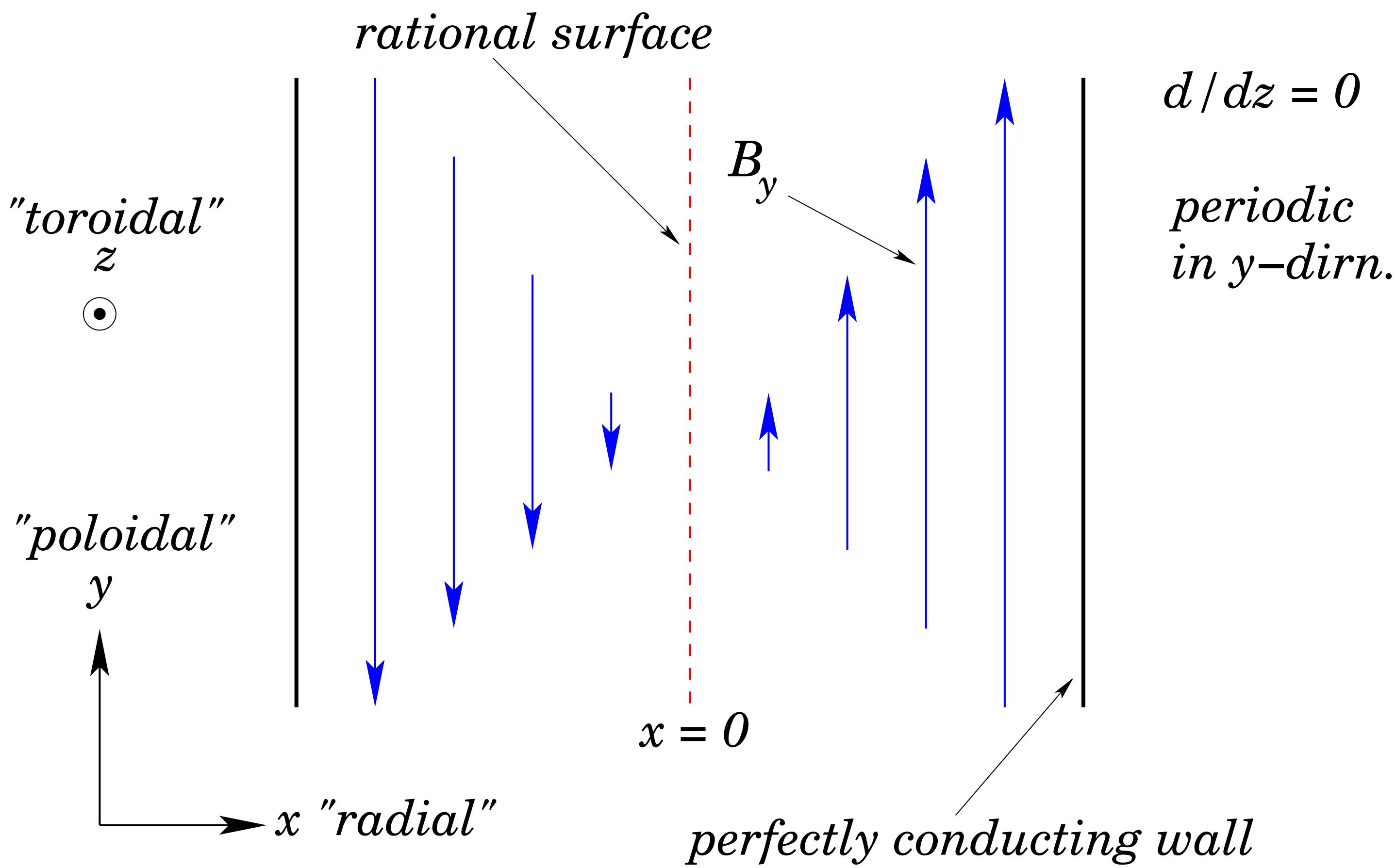
[a] *Plasma Confinement*, R.D. Hazeltine, and J.D. Meiss (Dover, 2003).

Slab Approximation



Slab Model

- Cartesian coordinates: (x, y, z). Let $\partial/\partial z \equiv 0$.
- Assume presence of dominant uniform "guide-field" $\vec{B}_z\,\vec{e}_z$.
- All field-strengths normalized to B_z.
- All lengths normalized to equilibrium magnetic shear-length:

$$L_s = B_z/(dB_y^{(0)}/dx)_{x=0}.$$

- All times normalized to Alfvén time calculated with B_z.
- Perfect wall boundary conditions at $x = \pm a$.
- Wave-number of tearing instability: $\vec{k} = (0, k, 0)$, so $\vec{k}\cdot\vec{B} = 0$ at $x = 0$. Hence, rational surface at $x = 0$.

Model MHD Equations

- Let $\vec{B} = \nabla\psi \times \vec{e}_z + B_z\,\vec{e}_z$, $\vec{V} = \nabla\phi \times \vec{e}_z$, where $\vec{V}$ is $\vec{E}\times\vec{B}$ vely.
- $\vec{B}\cdot\nabla\psi = \vec{V}\cdot\nabla\phi = 0$, so ψ maps magnetic flux-surfaces, and ϕ maps stream-lines of $\vec{E}\times\vec{B}$ fluid.
- Incompressible MHD equations: [a]

$$\frac{\partial\psi}{\partial t} = [\phi, \psi] + \eta\, J,$$

$$\frac{\partial U}{\partial t} = [\phi, U] + [J, \psi] + \mu\,\nabla^2 U,$$

where $J = \nabla^2\psi$, $U = \nabla^2\phi$, and $[A, B] = A_x\,B_y - A_y\,B_x$. Here, η is resistivity, and μ is viscosity. In normalized units: $\eta, \mu \ll 1$.

- First equation is z-component of Ohm's law. Second equation is z-component of curl of plasma equation of motion.

[a] *Plasma Confinement*, R.D. Hazeltine, and J.D. Meiss (Dover, 2003).

Outer Region

- In "outer region", which comprises most of plasma, can neglect non-linear, non-ideal (η and μ), and inertial ($\partial/\partial t$ and $\vec{V}\cdot\nabla$) effects.

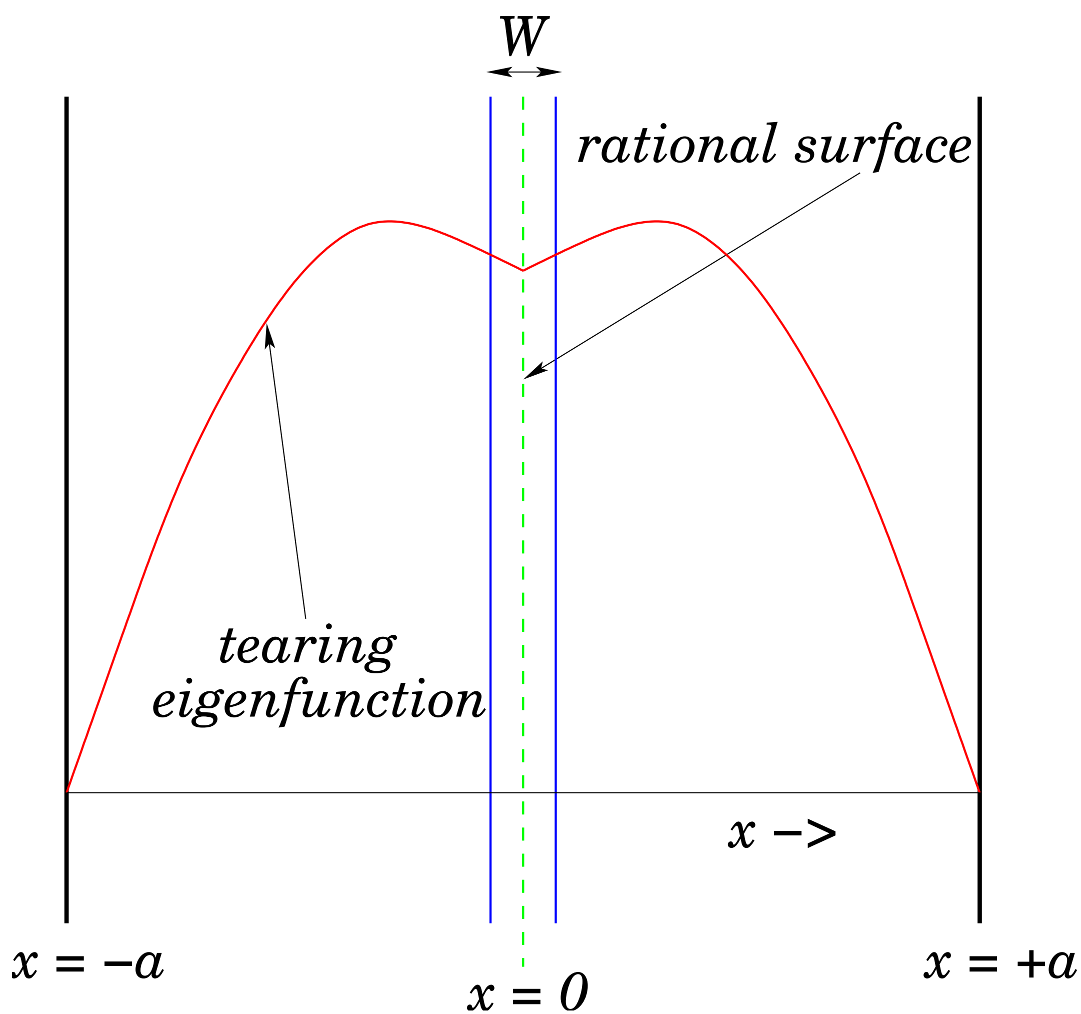
- Vorticity equation reduces to

$$[J, \psi] \simeq 0.$$

- When linearized, obtain $\psi(x, y) = \psi^{(0)}(x) + \psi^{(1)}(x)\,\cos(k\,y)$, where $B_y^{(0)} = -d\psi^{(0)}/dx$, and

$$\left(\frac{d^2}{dx^2} - k^2\right)\psi^{(1)} - \left(\frac{d^2 B_y^{(0)}/dx^2}{B_y^{(0)}}\right)\psi^{(1)} = 0.$$

- Equation is *singular* at rational surface, $x = 0$, where $B_y^{(0)} = 0$.



W
rational surface
tearing
eigenfunction
x ->
x = −a
x = 0
x = +a

Tearing Stability Index

- Find tearing eigenfunction, $\psi^{(1)}(x)$, which is continuous, has tearing parity $[\psi^{(1)}(-x) = \psi^{(1)}(x)]$, and satisfies boundary condition $\psi^{(1)}(a) = 0$ at conducting wall.
- In general, eigenfunction has *gradient discontinuity* across rational surface (at $x = 0$). Allowed because tearing mode equation singular at rational surface.
- Tearing stability index:

$$\Delta' = \left[\frac{d \ln \psi^{(1)}}{dx}\right]_{0-}^{0+}.$$

- According to conventional MHD theory,[a] tearing mode is unstable if $\Delta' > 0$.

[a] H.P. Furth, J. Killeen, and M.N. Rosenbluth, Phys. Fluids **6**, 459 (1963).

Inner Region

- "Inner region" centered on rational surface, $x = 0$. Of extent, $W \ll 1$, where W is magnetic island width (in x).
- In inner region, non-ideal effects, non-linear effects, and plasma inertia can all be important.
- Inner solution must be asymptotically matched to outer solution already obtained.

Constant-ψ Approximation

- $\psi^{(1)}(x)$ generally does not vary significantly in x over inner region:

$$|\psi^{(1)}(W) - \psi^{(1)}(0)| \ll |\psi^{1}(0)|.$$

- *Constant-ψ approximation*: treat $\psi^{(1)}(x)$ as constant in x over inner region.
- Approximation valid provided

$$|\Delta'|\, W \ll 1,$$

which is easily satisfied for conventional tearing modes in tokamak plasmas.

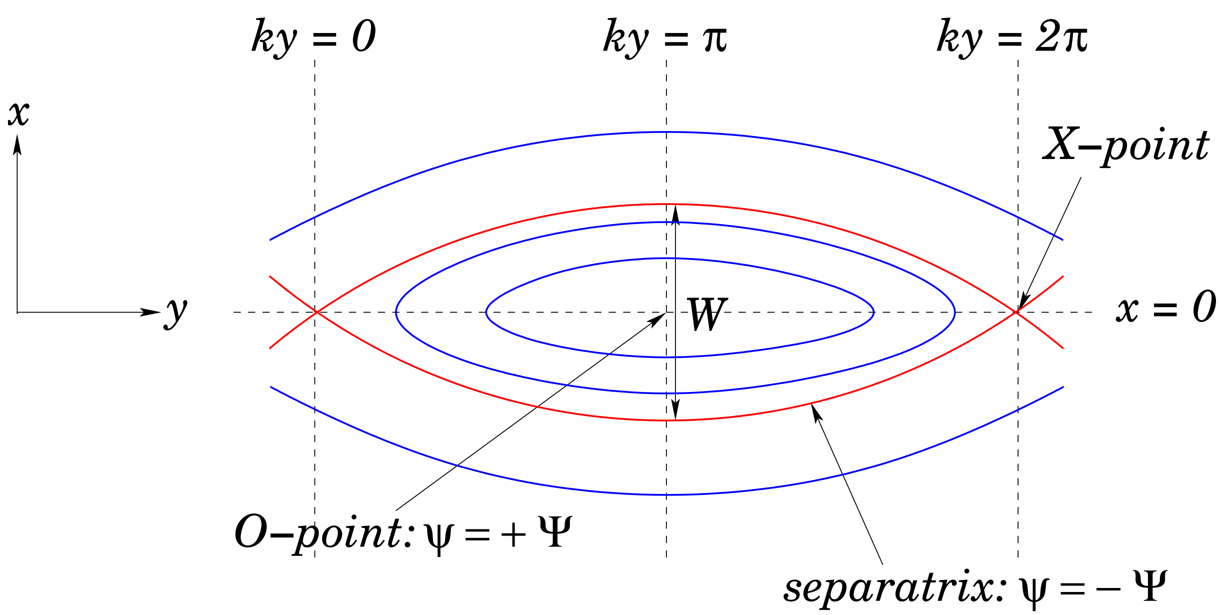
Constant-ψ Magnetic Island

- In vicinity of rational surface, $\psi^{(0)} \to -x^2/2$, so

$$\psi(x, y, t) \simeq -x^2/2 + \Psi(t) \cos\theta,$$

where $\Psi = \psi^{(1)}(0)$ is "reconnected flux", and $\theta = k\,y$.

- Full island width, $W = 4\sqrt{\Psi}$.



Flux-Surface Average Operator

- Flux-surface average operator is annihilator of Poisson bracket $[A, \psi] \equiv \vec{B} \cdot \nabla A \equiv k\, x\, (\partial A/\partial\theta)_\psi$ for any A: *i.e.*,

$$\langle [A, \psi] \rangle \equiv 0.$$

- Outside separatrix:

$$\langle f(\psi, \theta) \rangle = \oint \frac{f(\psi, \theta)}{|x|} \frac{d\theta}{2\pi}.$$

- Inside separatrix:

$$\langle f(s, \psi, \theta) \rangle = \int_{-\theta_0}^{\theta_0} \frac{f(s, \psi, \theta) + f(-s, \psi, \theta)}{2\,|x|} \frac{d\theta}{2\pi},$$

where $s = \mathrm{sgn}(x)$, and $x(s, \psi, \theta_0) = 0$.

MHD Flow - I

- Move to island frame. Look for steady-state solution: $\partial/\partial t = 0$.[a]
- Ohm's law:

$$0 \simeq [\phi, \psi] + \eta \, J.$$

- Since $\eta \ll 1$, first term potentially much larger than second.
- To lowest order:

$$[\phi, \psi] \simeq 0.$$

- Follows that

$$\phi = \phi(\psi):$$

i.e., MHD flow constrained to be around flux-surfaces.

[a] F.L. Waelbroeck, and R. Fitzpatrick, Phys. Rev. Lett. **78**, 1703 (1997).

MHD Flow - II

- Let

$$M(\psi) = \frac{d\phi}{d\psi}.$$

- Easily shown that

$$V_y = x\,M.$$

- By symmetry, $M(\psi)$ is *odd* function of x. Hence,

$$M = 0$$

inside separatrix: *i.e.*, no flow inside separatrix in island frame. Plasma *trapped* within magnetic separatrix.

MHD Flow - III

- Vorticity equation:

$$0 \simeq [-M\,U + J, \psi] + \mu\,\nabla^4\phi.$$

- Flux-surface average, recalling that $\langle [A, \psi]\rangle = 0$:

$$\langle \nabla^4\phi\rangle \equiv -\frac{d^2}{d\psi^2}\left(\langle x^4\rangle\,\frac{dM}{d\psi}\right) \simeq 0.$$

- Solution outside separatrix:

$$M(\psi) = \mathrm{sgn}(x)\,M_0 \int_{-\Psi}^{\psi} d\psi/\langle x^4\rangle \Bigg/ \int_{-\Psi}^{-\infty} d\psi/\langle x^4\rangle.$$

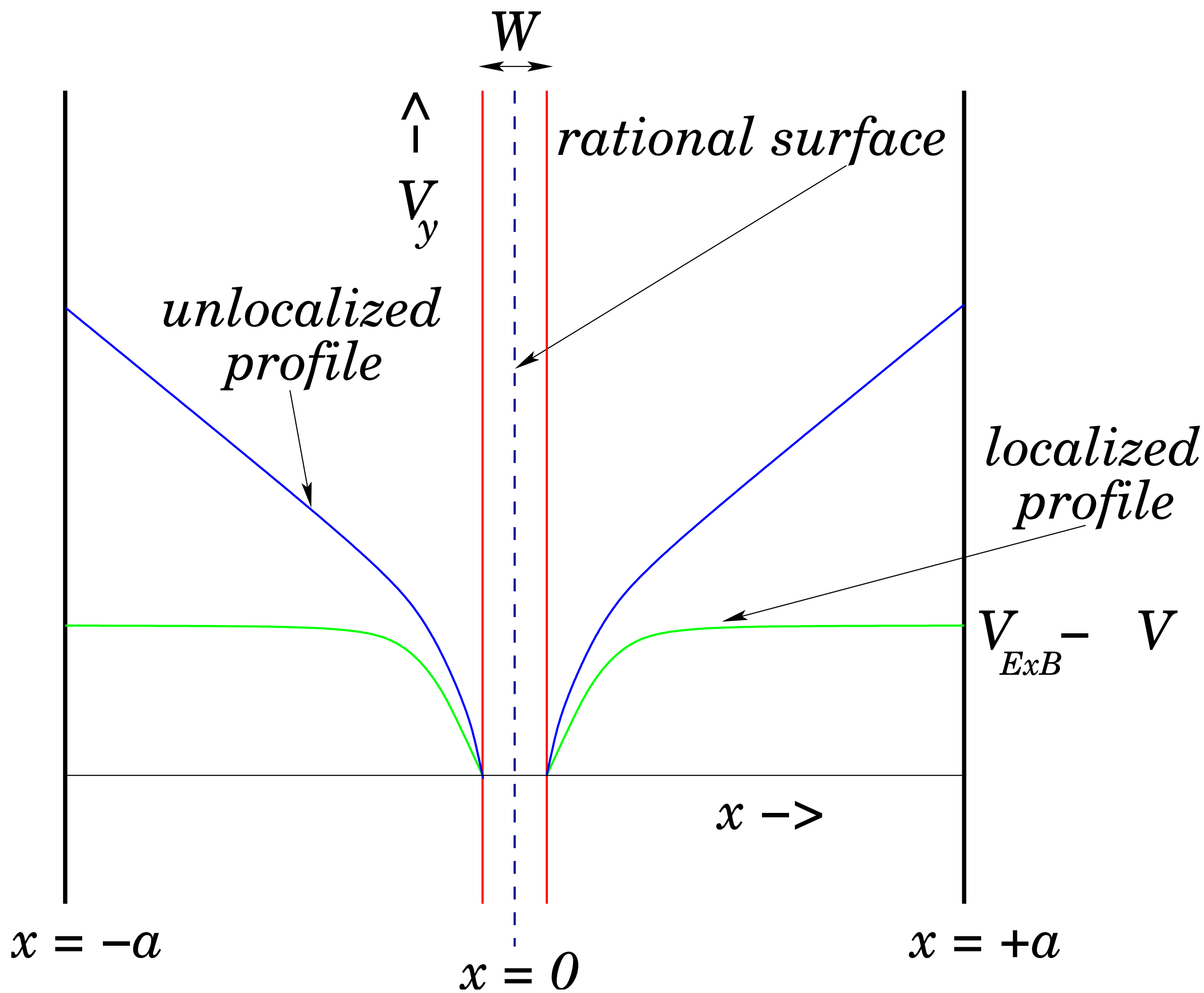
MHD Flow - IV

- Note

$$V_y = x\,M \rightarrow |x|\,M_0$$

  as $|x|/W \rightarrow \infty$.

- V-shaped velocity profile that extends over whole plasma.
- Expect *isolated* magnetic island to have *localized* velocity profile. Suggests that $M_0 = 0$ for isolated island.
- Hence, zero MHD flow in island frame: *i.e.*, island propagates at local $\vec{E} \times \vec{B}$ velocity.



W
rational surface
V_y
unlocalized profile
localized profile
$V_{ExB} - V$
x ->
x = −a
x = 0
x = +a

Rutherford Equation - I

- Asymptotic matching between inner and outer regions yields:

$$\Delta' \Psi = -4 \int_{+\Psi}^{-\infty} \langle J \cos\theta \rangle \, d\psi.$$

- In island frame, in absence of MHD flow, vorticity equation reduces to

$$[J, \psi] \simeq 0.$$

- Hence,

$$J = J(\psi).$$

Rutherford Equation - II

- Ohm's law:

$$\frac{d\Psi}{dt}\cos\theta \simeq [\phi, \psi] + \eta\, J(\psi).$$

- Have shown there is no MHD-flow [*i.e.*, $\phi \sim O(1)$], but can still be *resistive flow* [*i.e.*, $\phi \sim O(\eta)$].

- Eliminate resistive flow by flux-surface averaging:

$$\frac{d\Psi}{dt}\,\langle\cos\theta\rangle \simeq \eta\, J(\psi)\,\langle 1\rangle.$$

- Hence,

$$\Delta'\,\Psi \simeq -\frac{4}{\eta}\frac{d\Psi}{dt}\int_{+\Psi}^{-\infty}\frac{\langle\cos\theta\rangle^2}{\langle 1\rangle}\,d\psi.$$

Rutherford Equation - III

- Use $W = 4\sqrt{\Psi}$, and evaluate integral. Obtain *Rutherford island width evolution equation*: [a]

$$\frac{0.823}{\eta}\frac{dW}{dt} \simeq \Delta'.$$

- According to Rutherford equation, island grows *algebraically* on *resistive time-scale*.

- Rutherford equation does not predict island saturation.

[a] P.H. Rutherford, Phys. Fluids **16**, 1903 (1973).

Rutherford Equation - IV

- Higher order asymptotic matching between inner and outer regions yields: [a]

$$\frac{0.823}{\eta}\frac{dW}{dt} \simeq \Delta' - 0.41\left(-\frac{d^4B_y^{(0)}/dx^4}{d^2B_y^{(0)}/dx^2}\right)_{x=0} W.$$

- Hence, saturated $(d/dt = 0)$ island width is

$$W_0 = \frac{\Delta'}{0.41}\left(-\frac{d^2B_y^{(0)}/dx^2}{d^4B_y^{(0)}/dx^4}\right)_{x=0}.$$

[a] F. Militello, and F. Porcelli, Phys. Plasmas **11**, L13 (2004). D.F. Escande, and M. Ottaviani, Physics Lett. A **323**, 278 (2004).

Main Predictions of MHD Theory

- Tearing mode unstable if $\Delta' > 0$.
- Island propagates at local $\vec{E} \times \vec{B}$ velocity at rational surface.
- Island grows algebraically on resistive time-scale.
- Saturated island width:

$$W_0 = \frac{\Delta'}{0.41} \left(- \frac{d^2 B_y^{(0)}/dx^2}{d^4 B_y^{(0)}/dx^4} \right)_{x=0}.$$

4: Profile Modification

Temperature Flattening - I [a]

- For sake of simplicity, assume cold ions. Let T be electron temperature profile (minus uniform background value, T_e).
- In immediate vicinity of island, unperturbed profile is

$$T(x) \simeq T_e\,\frac{x}{L_T},$$

where L_T is equilibrium temperature gradient scale-length.

- Perturbed temperature profile determined by competition between parallel and perpendicular heat transport:

$$\chi_\parallel\,\nabla_\parallel^{\,2} T + \chi_\perp\,\nabla_\perp^{\,2}\,T \simeq 0.$$

[a] R. Fitzpatrick, Phys. Plasmas **2**, 825 (1995).

Temperature Flattening - II

- Parallel transport term attempts to make temperature a flux-surface function. Cannot have odd flux-surface function inside island separatrix. So, if $T = T(\psi)$ then temperature is flattened inside separatrix.
- Perpendicular transport term attempts to relax temperature profile to unperturbed profile. Opposes temperature flattening.
- Temperature is flattened inside separatrix if parallel transport term dominates perpendicular transport term.

Temperature Flattening - III

- Have

$$\nabla_{\parallel} \simeq \frac{k\,W}{L_s}.$$

- Also

$$\nabla_{\perp} \simeq \frac{1}{W}.$$

- Hence, parallel transport dominates perpendicular transport when

$$W \gg W_c,$$

where

$$k\,W_c \sim \left(\frac{\chi_{\perp}}{\chi_{\parallel}}\right)^{1/4} (k\,L_s)^{1/2}.$$

Temperature Flattening - IV

- In "collisionless" fusion plasmas, Braginskii[a] expression for parallel heat flux due to conduction impossibly large: *i.e.*,

$$\chi_\parallel \nabla_\parallel T \gg n_e v_e T,$$

where n_e is uniform background electron number density, and v_e is thermal velocity.

- In this situation, parallel heat transport becomes *convective* in nature, rather than *diffusive*. Can model this effect by "flux limiting" heat flux: *i.e.*, replace $\chi_\parallel \nabla_\parallel T$ by $n_e v_e T$.
- Leads to more realistic expression for critical island width[b]

$$k W_c \sim \left(\frac{\chi_\perp}{n_e v_e L_s}\right)^{1/3} (k L_s)^{2/3}.$$

[a]S.I. Braginskii, Reviews of Plasma Physics, (Consultants Bureau, 1965)
[b]N.N. Gorolenkov, *et al.*, Phys. Plasmas **3**, 3379 (1996).

Temperature Flattening - IV

- Temperature profile only flattened inside separatrix when island width exceeds critical value.
- Critical width much smaller than minor radius in conventional tokamak.
- Temperature flattening implies complete loss of radial energy confinement across island.

Density Flattening - I [a]

- Let n be electron number density (minus uniform background value, n_e).
- In immediate vicinity of magnetic island, unperturbed density profile is
$$n(x) \simeq n_e \frac{x}{L_n},$$
where L_n is equilibrium density gradient scale-length.
- Sound waves propagating along magnetic field-lines act to make density a flux-surface function. Cannot have odd flux-surface function inside island separatrix. So, if $n = n(\psi)$ then density is flattened inside separatrix.

[a] R. Fitzpatrick, Phys. Plasmas **2**, 825 (1995).

Density Flattening - II

- Expect sound waves to flatten density profile when [a]

$$(\vec{V}_* \cdot \nabla)\, n \ll c_s\, \nabla_{\|} n,$$

where $c_s = \sqrt{T_e/m_i}$ is (unnormalized) *sound speed*, and

$$\vec{V}_* = c_s\, \frac{\rho}{L_n}\, \vec{e}_y$$

is *diamagnetic velocity* due to equilibrium density gradient. Furthermore, $\rho = c_s/(e\, B_z/m_i)$ is ion Larmor radius calculated with electron temperature.

[a] A.I. Smolyakov, Plasma Phys. Control. Fusion **35**, 657 (1993).

Density Flattening - III

- Density profile flattened when island width exceeds critical value

$$W_c \sim \rho\,\frac{L_s}{L_n}.$$

- In typical tokamak plasma, critical width for density flattening generally considerably larger than that for temperature flattening, but still much smaller than minor radius.

- If island width exceeds critical value for density flattening then pressure profile completely flattened inside island separatrix. Implies complete loss of radial energy and particle confinement across island.

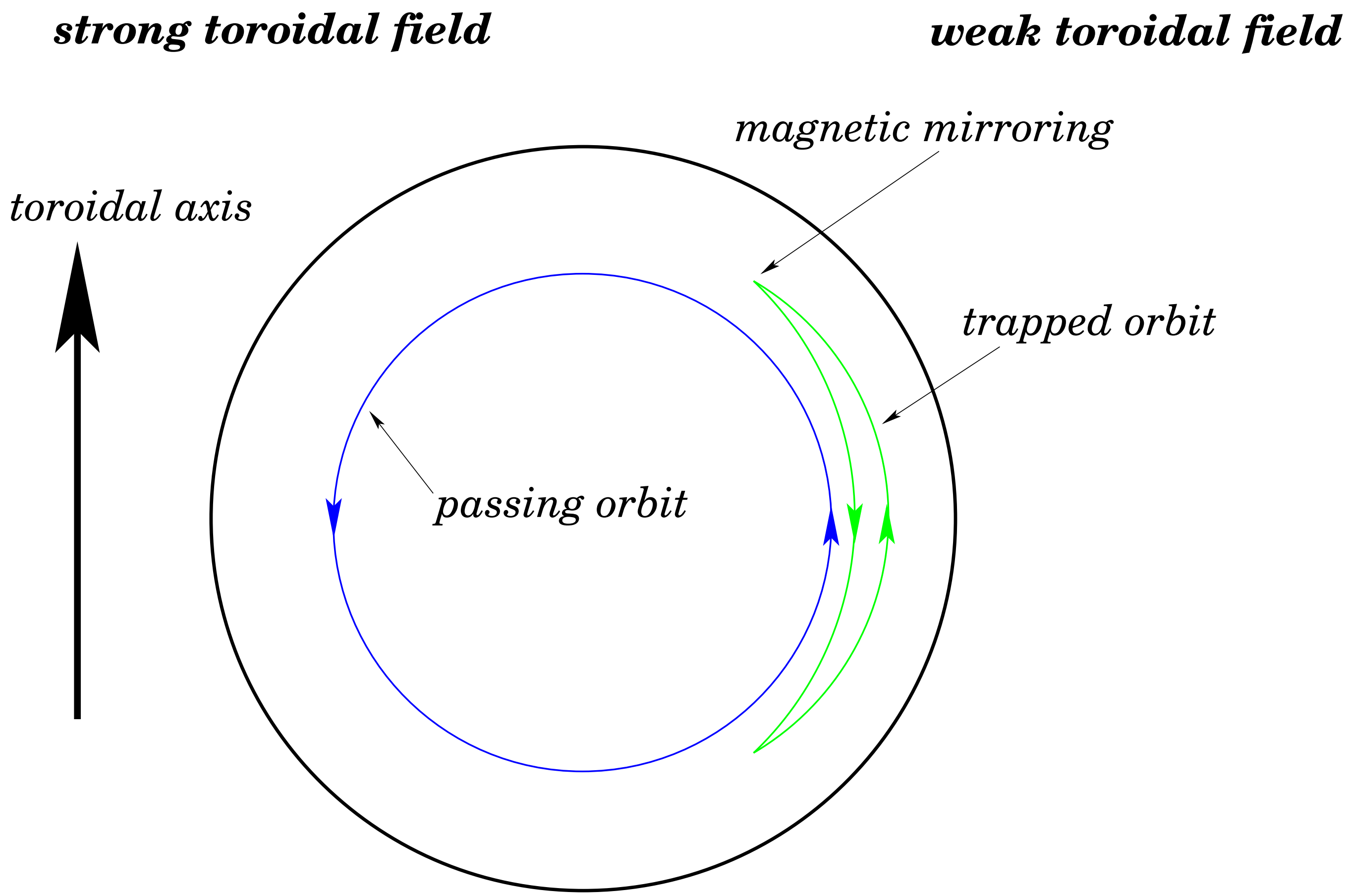
5: Bootstrap Current Destabilization

Neoclassical Effects

- So-called *neoclassical effects*[a] in tokamak plasmas arise from essential *toroidicity* of such plasmas, combined with extremely long mean-free-path of electrons and ions streaming along magnetic field-lines, due to high plasma temperature.

[a] *The Theory of Toroidally Confined Plasmas*, 2nd Rev. Edition, R.B. White (World Scientific, 2006).

Trapped and Passing Particles



Bootstrap Current - I

- In toroidal plasma, friction between trapped and passing electrons leads to appearance of non-inductive *bootstrap current* in Ohm's law: [a]

$$\frac{d\Psi}{dt}\cos\theta \simeq [\phi,\psi] + \eta\,[J(\psi) - J_{\rm boot}],$$

where

$$J_{\rm boot} = -1.46\,\sqrt{\epsilon}\,B_\theta^{-1}\,\frac{dP}{dx}.$$

Here, ϵ is inverse aspect-ratio, $1.46\,\sqrt{\epsilon}$ is measure of fraction of trapped-particles, B_θ is poloidal magnetic field-strength, and P is plasma pressure.

[a] M.N. Rosenbluth, R.D. Hazeltine, and F.L. Hinton, Phys. Fluids **15**, 116 (1972).

Bootstrap Current - II

- For sufficiently wide island, pressure profile *flattened* inside separatrix.
- Bootstrap current consequently disappears inside separatrix.
- Absence of bootstrap current inside separatrix, and continued presence outside, leads to *destabilizing* term in Rutherford island equation: [a]

$$\frac{0.823}{\eta}\frac{dW}{dt} \simeq \Delta' + 9.25\,\sqrt{\epsilon}\,\beta\,\frac{L_s}{L_p}\,\frac{B_z}{B_\theta}\,\frac{1}{W},$$

where $\beta = \mu_0\, n_e\, T_e/B_z^2$, and $L_p^{-1} = L_n^{-1} + L_T^{-1}$.

[a] R. Carrera, R.D. Hazeltine, and M. Kotschenreuther, Phys. Fluids **29**, 899 (1986).

Neoclassical Tearing Modes

- A *neoclassical tearing mode* (NTM) is an *intrinsically stable* ($\Delta' < 0$) tearing mode destabilized by bootstrap term.
- Bootstrap term in Rutherford equation relatively large, especially at small island widths. Would expect plasma to be filled with NTMs, and confinement to be wrecked.[a]
- This is not observed to be case. Experimental evidence for threshold island width above which NTMs grow, but below which they decay.[b]
- Suggests presence of stabilizing effect in Rutherford equation that opposes destabilizing bootstrap term.

[a] C.C. Hegna, J.D. Callen, Phys. Fluids B **4**, 1855 (1992).
[b] O. Sauter, *et al.*, Phys. Plasmas **4**, 1654 (1997).

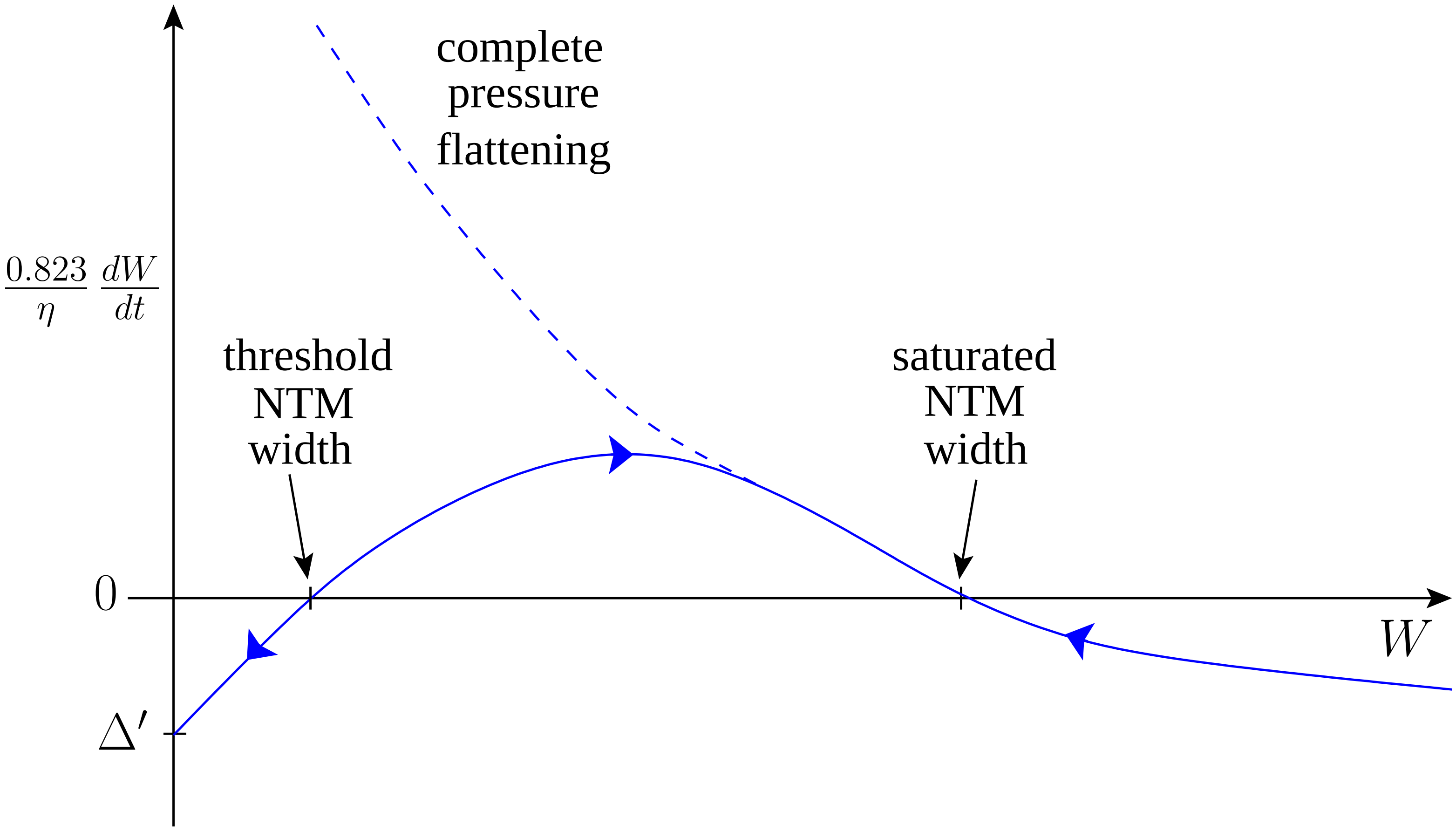
Incomplete Pressure Flattening - I

- Bootstrap destabilization is caused by flattening of pressure profile inside island separatrix. If there is no flattening then there is no destabilization.
- Pressure flattening only occurs when island width exceeds critical value W_c.
- When incomplete pressure flattening incorporated into Rutherford equation find that [a]

$$\frac{0.823}{\eta}\frac{dW}{dt} \simeq \Delta' + 9.25\sqrt{\epsilon}\,\beta\,\frac{L_s}{L_P}\,\frac{B_z}{B_\theta}\,\frac{W}{W^2+W_c^2}.$$

[a] R. Fitzpatrick, Phys. Plasmas **2**, 825 (1995).

Incomplete Pressure Flattening - II



6: Drift-MHD Theory

Introduction

- In drift-MHD model, which is far more accurate representation of tokamak plasma than MHD model, analysis retains *charged particle drift velocities*, in addition to $\vec{E} \times \vec{B}$ velocity.
- Essentially two-fluid theory of plasma.
- Characteristic length-scale, ρ, is *ion Larmor radius calculated with electron temperature*.
- Characteristic velocity is diamagnetic velocity, V_*.
- Normalize all lengths to ρ, and all velocities to V_*.

Basic Assumptions

- Retain slab model, for sake of simplicity.
- Assume island sufficiently wide that $T = T(\psi)$.
- Assume $T_i/T_e = \tau = \text{constant}$, for sake of simplicity, where T_i and T_e are complete electron and ion temperature profiles.

Basic Definitions

- Variables:
  - ψ - magnetic flux-function.
  - J - parallel current density.
  - ϕ - guiding-center (*i.e.*, MHD) stream-function.
  - U - parallel ion vorticity.
  - n - electron number density (minus uniform background).
  - V_z - parallel ion velocity.
- Parameters:
  - $\alpha = (L_n/L_s)^2$, where L_s is magnetic shear length, and L_n is density gradient scale-length.
  - η - resistivity. D - (perpendicular) particle diffusivity. $\mu_{i/e}$ - (perpendicular) ion/electron viscosity.

Drift-MHD Equations - I

- Steady-state drift-MHD equations: [a]

$$
\begin{aligned}
\psi &= -x^2/2 + \Psi \cos\theta, \qquad U = \nabla^2\phi, \\
0 &= [\phi - n, \psi] + \eta\, J, \\
0 &= [\phi, U] - \frac{\tau}{2}\left\{\nabla^2[\phi, n] + [U, n] + [\nabla^2 n, \phi]\right\} \\
&\quad +[J, \psi] + \mu_i \nabla^4(\phi + \tau\, n) + \mu_e\, \nabla^4(\phi - n), \\
0 &= [\phi, n] + [V_z + J, \psi] + D\, \nabla^2 n, \\
0 &= [\phi, V_z] + \alpha\, [n, \psi] + \mu_i\, \nabla^2 V_z.
\end{aligned}
$$

[a] R.D. Hazeltine, M. Kotschenreuther, and P.J. Morrison, Phys. Fluids **28**, 2466 (1985).

Drift-MHD Equations - II

- Symmetry: ψ, J, V_z even in x. ϕ, n, U odd in x.
- Boundary conditions as $|x|/W \rightarrow \infty$:
  - $n \rightarrow -(1+\tau)^{-1}\, x$.
  - $\phi \rightarrow -V\, x$.
  - $J, U, V_z \rightarrow 0$.
- Here, V is island phase-velocity in $\vec{E} \times \vec{B}$ frame.
- $V = 1$ corresponds to island propagating with electron fluid. $V = -\tau$ corresponds to island propagating with ion fluid.
- Expect

$$1 \gg \alpha \gg \eta, D, \mu_i, \mu_e.$$

Electron Fluid

- Ohm's law:

$$0 = [\phi - n, \psi] + \eta\, J.$$

- Since $\eta \ll 1$, first term potentially much larger than second.
- To lowest order:

$$[\phi - n, \psi] \simeq 0.$$

- Follows that

$$n = \phi + H(\psi):$$

i.e., electron stream-function $\phi_e = \phi - n$ is *flux-surface function*. Electron fluid flow constrained to be around flux-surfaces.

Sound Waves

- Parallel flow equation:

$$0 = [\phi, V_z] + \alpha\,[n, \psi] + \mu_i\,\nabla^2 V_z.$$

- Highlighted term dominant provided

$$W \gg \alpha^{-1/2} = L_s/L_n.$$

- If this is case then, to lowest order,

$$n = n(\psi),$$

which implies $n = 0$ inside separatrix.

- So, if island sufficiently wide then *sound-waves* able to *flatten density profile* inside island separatrix.

Subsonic vs. Supersonic Islands

- Wide islands satisfying

$$W \gg L_s/L_n$$

termed *subsonic* islands. Expect such islands to exhibit flattened density profile within separatrix. Subsonic islands strongly coupled to both electron and ion fluids.

- Narrow islands satisfying

$$W \ll L_s/L_n$$

termed *supersonic* islands. No flattening of density profile within separatrix. Supersonic islands strongly coupled to electron fluid, but only weakly coupled to ion fluid.

7: Subsonic Island Theory

Introduction[a]

- To lowest order:

$$\phi = \phi(\psi), \;\; n = n(\psi).$$

- Follows that both electron stream-function, $\phi_e = \phi - n$, and ion stream-function, $\phi_i = \phi + \tau\, n$, are flux-surface functions. Both electron and ion fluid flow constrained to follow flux-surfaces.

- Let

$$M(\psi) = d\phi/d\psi, \;\; L(\psi) = dn/d\psi.$$

- Follows that

$$V_{E\times B\,y} = x\, M, \;\; V_{e\,y} = x\,(M - L), \;\; V_{i\,y} = x\,(M + \tau\, L).$$

[a] R. Fitzpatrick, F.L. Waelbroeck, Phys. Plasmas **12**, 022307 (2005).

Density Flattening

- By symmetry, both $M(\psi)$ and $L(\psi)$ are *odd* functions of x. Hence,

$$M(\psi) = L(\psi) = 0$$

  inside separatrix: *i.e.*, no electron/ion flow within separatrix in island frame.

- Electron/ion fluids constrained to propagate with island inside separatrix.

- Density profile *flattened* within separatrix.

Analysis - I

- Density equation reduces to

$$0 \simeq [V_z + J, \psi] + D\,\nabla^2 n.$$

- Vorticity equation reduces to

$$\begin{aligned} 0 &\simeq \left[-M\,U - (\tau/2)(L\,U + M\,\nabla^2 n) + J, \psi\right] \\ &\quad + \mu_i\,\nabla^4(\phi + \tau\,n) + \mu_e\,\nabla^4(\phi - n). \end{aligned}$$

- Flux-surface average both equations, recalling that $\langle [A, \psi] \rangle = 0$.

Analysis - II

- Obtain

$$\langle \nabla^2 n \rangle \simeq 0,$$

and

$$(\mu_i + \mu_e)\,\langle \nabla^4 \phi \rangle + (\mu_i\,\tau - \mu_e)\,\langle \nabla^4 n \rangle \simeq 0.$$

- Solution outside separatrix:

$$M(\psi) = -\frac{(\mu_i\,\tau - \mu_e)}{(\mu_i + \mu_e)}\,L(\psi) + F(\psi),$$

where

$$L(\psi) = -\mathrm{sgn}(x)\,L_0/\langle x^2 \rangle,$$

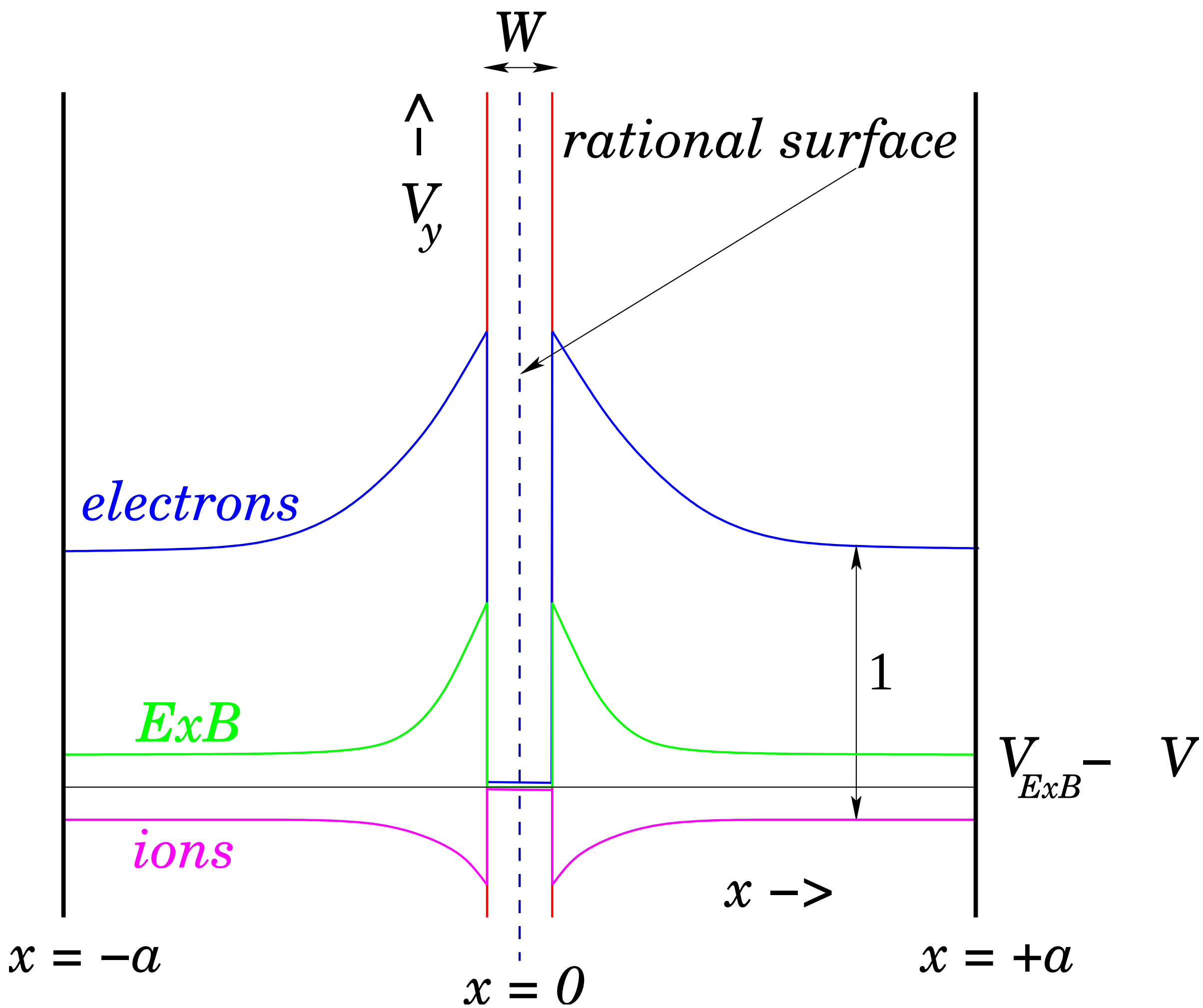
and $F(\psi)$ is previously obtained MHD profile:

$$F(\psi) = \mathrm{sgn}(x)\,F_0\,\int_{-\Psi}^{\psi} d\psi/\langle x^4 \rangle \bigg/ \int_{-\Psi}^{-\infty} d\psi/\langle x^4 \rangle.$$

Velocity Profiles

- As $|x|/W \to \infty$ then $x\,L \to L_0$ and $x\,F \to |x|\,F_0$.
- $L(\psi)$ corresponds to *localized* velocity profile. $F(\psi)$ corresponds to *non-localized* profile. Require localized profile, so $F_0 = 0$.
- Velocity profiles outside separatrix (using b.c. on n):

$$\begin{aligned} V_{y\,i} &\simeq +\frac{\mu_e}{\mu_i + \mu_e}\,\frac{|x|}{\langle x^2\rangle}, \\ V_{y\,E\times B} &\simeq -\frac{(\mu_i\,\tau - \mu_e)}{(1+\tau)\,(\mu_i + \mu_e)}\,\frac{|x|}{\langle x^2\rangle}, \\ V_{y\,e} &= -\frac{\mu_i}{\mu_i + \mu_e}\,\frac{|x|}{\langle x^2\rangle}. \end{aligned}$$



W
^
V_y
rational surface
electrons
ExB
1
V_ExB − V
ions
x ->
x = −a
x = 0
x = +a

Island Propagation

- As $|x|/W \to \infty$ expect $V_{y\,E\times B} \to V_{EB} - V$, where V_{EB} is unperturbed (*i.e.*, no island) $\vec{E} \times \vec{B}$ velocity at rational surface (in lab. frame), and V is island phase-velocity (in lab. frame).

- Hence
$$V = V_{EB} + \frac{(\mu_i\,\tau - \mu_e)}{(1+\tau)\,(\mu_i + \mu_e)}.$$

- But unperturbed ion/electron fluid velocities (in lab. frame):
$$V_i = V_{EB} + \tau/(1+\tau), \quad V_e = V_{EB} - 1/(1+\tau).$$

- Hence
$$V = \frac{\mu_i}{\mu_i + \mu_e}\,V_i + \frac{\mu_e}{\mu_i + \mu_e}\,V_e.$$
So, island phase-velocity is *viscosity weighted average* of unperturbed ion/electron fluid velocities.

Polarization Term - I

- Vorticity equation yields

$$J_c \simeq \frac{1}{2}\left(x^2 - \frac{\langle x^2\rangle}{\langle 1\rangle}\right)\frac{d[M\,(M+\tau\,L)]}{d\psi} + I(\psi)$$

outside separatrix, where J_c is part of J with $\cos\theta$ symmetry.

- As before, flux-surface average of Ohm's law yields:

$$\langle J_c\rangle = I(\psi)\langle 1\rangle = \eta^{-1}\frac{d\Psi}{dt}\langle\cos\theta\rangle.$$

- Hence

$$J_c \simeq \frac{1}{2}\left(x^2 - \frac{\langle x^2\rangle}{\langle 1\rangle}\right)\frac{d[M\,(M+\tau\,L)]}{d\psi} + \eta^{-1}\frac{d\Psi}{dt}\frac{\langle\cos\theta\rangle}{\langle 1\rangle}.$$

Polarization Term - II

- Asymptotic matching between inner and outer regions yields:

$$\Delta' \Psi = -4 \int_{+\Psi}^{-\infty} \langle J_c \cos\theta \rangle \, d\psi.$$

- Evaluating flux-surface integrals, making use of previous solutions for M and L, obtain modified Rutherford equation:

$$\frac{0.823}{\eta} \frac{dW}{dt} \simeq \Delta' + 5.52 \, \beta \, (V - V_{EB}) \, (V - V_i) \, \frac{L_s^2}{L_n^2} \, \frac{1}{W^3}.$$

- New term is due to *polarization current* associated with ion fluid flow around curved island flux-surfaces (in island frame). Obviously, new term is zero if island propagates with ion fluid: *i.e.*, $V = V_i$.

Main Predictions of Subsonic Island Theory

- Results limited to large islands: *i.e.*, large enough for sound waves to flatten density profile.
- Island propagates at (perpendicular) viscosity weighted average of unperturbed (no island) ion and electron fluid velocities.
- Polarization term in Rutherford equation is stabilizing provided ion (perpendicular) viscosity greatly exceeds electron (perpendicular) viscosity (which is what we expect), and destabilizing otherwise.
- Polarization term ($\propto W^{-3}$) dominates bootstrap term ($\propto W^{-1}$) at small island widths, and *vice versa* at large island widths. Thus, polarization term can also provide threshold effect that prevents NTMs from growing until they exceed critical island width.

8: Supersonic Island Theory

Drift-MHD Equations [a]

- Steady-state drift-MHD equations (with $\tau = 0$, since ion diamagnetic effects largely irrelevant to supersonic islands):

$$
\begin{aligned}
\psi &= -x^2/2 + \Psi\cos\theta, \qquad U = \nabla^2\phi, \\
0 &= [\phi - n, \psi] + \eta\, J, \\
0 &= [\phi, U] + [J, \psi] + \mu_i\, \nabla^4\phi, \\
0 &= [\phi, n] + [V_z + J, \psi] + D\,\nabla^2 n, \\
0 &= [\phi, V_z] + \alpha\,[n, \psi] + \mu_i \nabla^2 V_z.
\end{aligned}
$$

[a] R. Fitzpatrick, and F.L. Waelbroeck, Phys. Plasmas **14**, 122502 (2007).

Zero-α Solution

- By definition, highlighted term small for supersonic islands.
- If term completely neglected, obtain trivial solution:

$$\phi = n = -x, \qquad U = V_z = J = 0.$$

- Island propagates with *electron fluid*.
- Island does not perturb ion fluid, so *zero polarization current*.

Small-α Solution

- Assume that highlighted term small, but not negligible. Perturb about zero-α solution.
- So

$$\phi = -x + \delta\phi, \qquad n = -x + \delta n,$$

where $\delta\phi$, δn, U, V_z, J all $O(\alpha) \ll 1$.

Analysis - I

- Lowest order solution:

$$\begin{aligned} \delta n &= \delta\phi + H(\psi), \\ J &= -\widetilde{G} + (\alpha/2)\,\widetilde{x^2}, \\ V_z &= -\alpha\,(W/4)^2\,\cos\theta, \end{aligned}$$

where $\widetilde{A} \equiv A - \langle A\rangle/\langle 1\rangle$.

- Here, $G = -x\,H'$. Now, $G = 0$ inside separatix, but outside separatrix

$$G = |x|\left(\frac{\langle x\,v\rangle + \alpha\,(W/4)^4}{\langle x^2\rangle}\right),$$

where $v = -\delta\phi_x$.

Analysis - II

- Perturbed velocity v satisfies

$$v_{xx} = (D/\mu)\,(\overline{v} - \overline{G}) - (G - \overline{G}) - \alpha\,(W/4)^2\,\cos\theta,$$

where $\overline{\cdots}$ denotes a θ-average at constant x.

- Boundary conditions: $v_x = 0$ at $x = 0$, and

$$v \rightarrow v_i + v_i'\,|x| - (\alpha/2)\,(W/4)^2\,x^2\,\cos\theta$$

as $|x| \rightarrow \infty$.

- Above equation highly nonlinear, but can be solved via iteration.

Need for Intermediate Layer

- Inner region island solution does not satsify $J \to 0$ as $|x| \to \infty$: *i.e.*, it does not asymptote to ideal-MHD solution in outer region.
- Require *intermediate layer* between island and outer region to allow proper matching.
- Intermediate layer much wider than island, so governed by linear physics.

Intermediate Layer - I

- Write

$$\phi(x,\theta) = -x + \overline{\delta\phi}(x) + \breve{\phi}(x)\,e^{\,i\,\theta}.$$

- Neglect all transport terms except ion viscosity.
- Linearized drift-MHD equations yield

$$\begin{aligned} \breve{\phi}_{xx} &- \bar{v}_{xx}\,\breve{\phi} - \left(\overline{v} - \frac{\alpha\,x^2}{1 - i\,\mu_i\,\alpha\,x^2}\right)\breve{\phi} \\ &= -\left(\overline{v} - \frac{\alpha\,x^2}{1 - i\,\mu_i\,\alpha\,x^2}\right)\frac{(W/4)^2}{x}, \end{aligned}$$

where $\overline{v} = -\overline{\delta\phi}_x$.

Intermediate Layer - II

- Mean velocity profile determined by *quasi-linear force balance*:

$$\bar{v}_{xx} = \frac{1}{2}\frac{\alpha^2\,x^2}{1+(\mu_i\,\alpha\,x^2)^2}\,|(W/4)^2 - x\,\breve{\phi}|^2.$$

- Perturbed current:

$$\breve{J} = (\breve{\phi}_{xx} - \bar{v}_{xx}\,\breve{\phi})/x.$$

Intermediate Layer - III

- Boundary conditions as $x \to 0$:

$$\breve{\phi} \to 0,$$
$$\bar{v} \to v_i + v_i' |x|.$$

- Boundary conditions as $|x| \to \infty$:

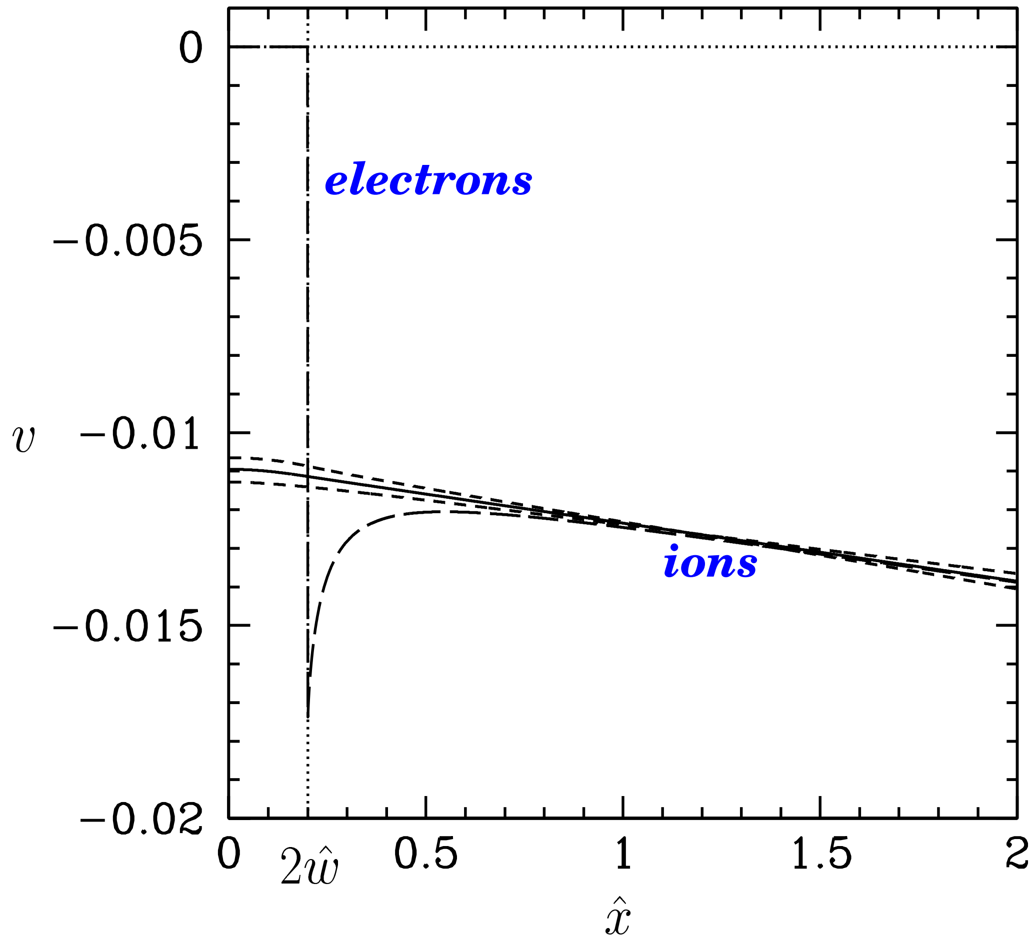
$$\breve{\phi} \to (W/4)^2/x,$$
$$\bar{v} \to v_\infty + v_\infty' |x|.$$

- Large-$|x|$ boundary conditions ensure that $\breve{J} \to 0$. So solution matches to ideal-MHD solution.

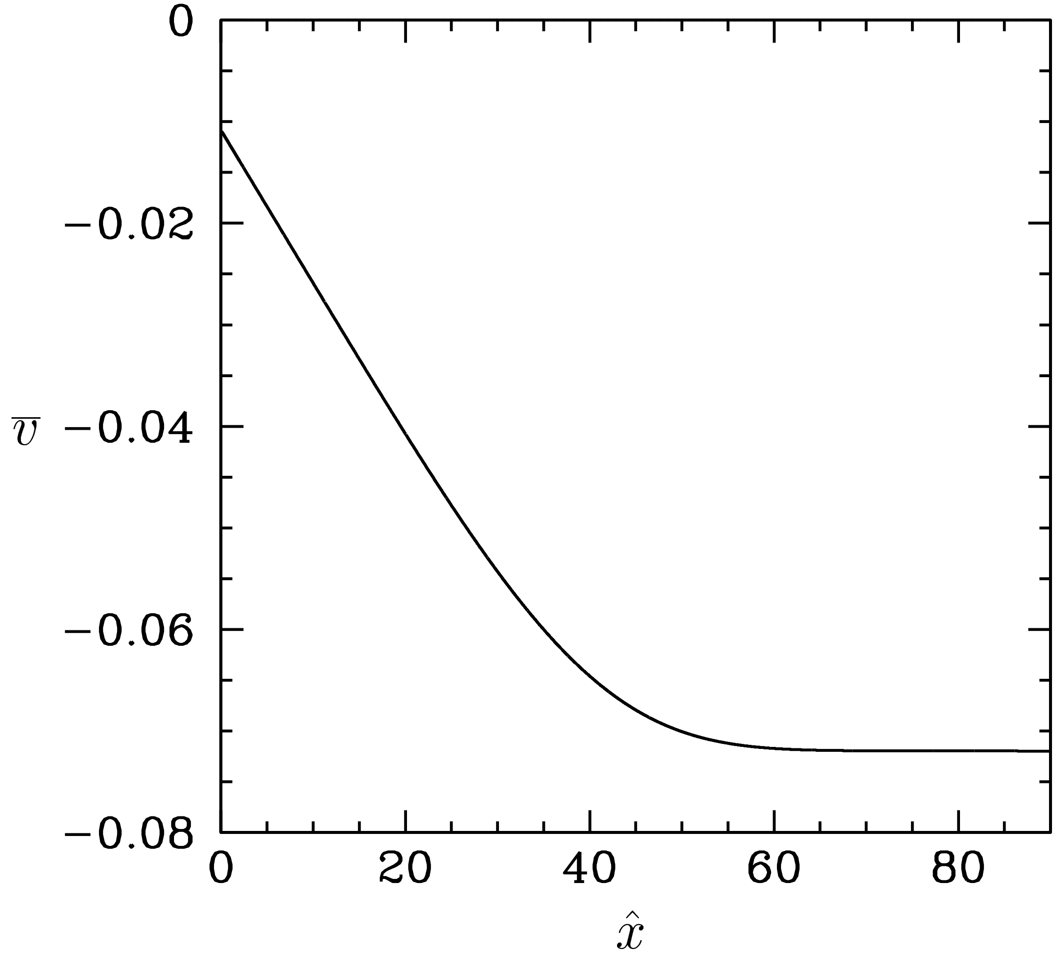
Physics of Intermediate Layer

- Island launches *drift-acoustic waves* into intermediate layer.
- Waves are *absorbed* in layer (due to ion viscosity).
- Waves carry *momentum*.
- Momentum exchange between island and intermediate layer ensures that velocity gradient, v_i', at inner boundary of layer not same as gradient, v_∞', at outer boundary.
- For isolated island solution, require $v_\infty' = 0$. This boundary condition *uniquely specifies* solution for given values of α, μ_i, D, etc.

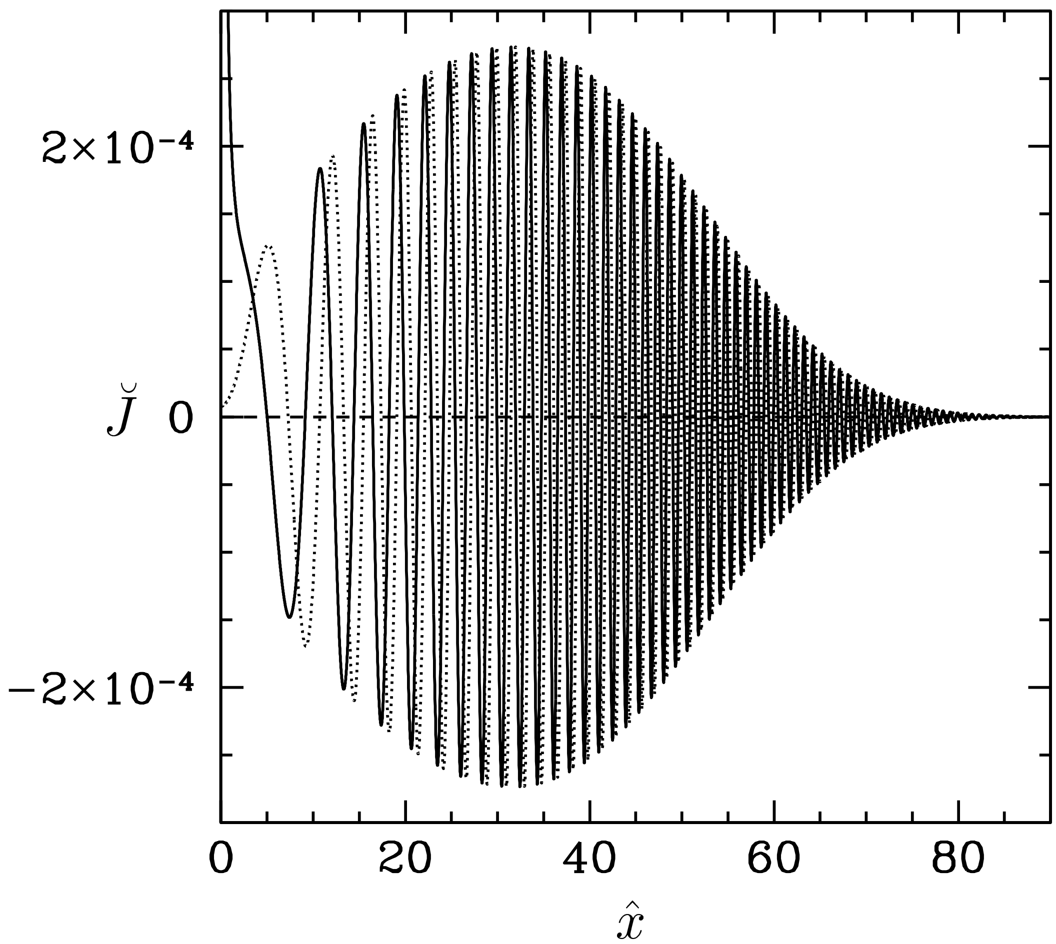
Velocity in Island Region



Velocity in Intermediate Layer



Current in Intermediate Layer



Island Propagation

- Island propagation velocity:

$$V = V_e - 0.27\,(W/4)^3\,\alpha^{3/4}\,D^{-1} - 0.24\,(W/4)^4\,\alpha^{1/3}\,\mu^{-4/3}.$$

- Island phase velocity close to unperturbed electron fluid velocity, but dragged slightly in ion direction due to sound-wave effects.

Ion Polarization Term

- Rutherford Equation:

$$\frac{0.823}{\eta}\,\frac{dW}{dt} = \Delta' - \frac{\beta}{\alpha^{1/4}}\,\frac{L_s^2}{L_n^2}\left[1.5 + 0.38\,(W/4)^2\,D^{-1}\right].$$

- Sound-wave effects ensure ion fluid slightly perturbed by island, generating polarization term in Rutherford equation. Term is *stabilizing*.

Maximum Island Width

- Supersonic branch of solutions ceases to exist beyond *maximum island width*:

$$W_{max} = 0.36\,\alpha^{-1/12}\,D^{1/3}.$$

- Hypothesized that island bifurcates to subsonic solution branch when $W > W_{max}$. This type of behavior has been observed in computer simulations.[a]

[a] M. Ottaviani, F. Porcelli, and D. Grasso, Phys. Rev. Lett. **93**, 075001 (2004).

Main Predictions of Supersonic Island Theory

- Results limited to small islands: *i.e.*, small enough that sound waves cannot flatten density profile.
- Islands phase velocities close to unperturbed electron fluid velocity, but dragged slightly in ion direction by sound wave effects.
- Islands radiate drift-acoustic waves.
- Momentum carried by drift-acoustic waves gives rise to strong velocity shear in region surrounding islands.
- Polarization term in Rutherford island equation is stabilizing.
- Supersonic branch ceases to exist above critical island width.

9: Other Effects

Neoclassical Flow Damping

- Poloidal (and, sometimes, toroidal) flow strongly damped in low-collisionality plasmas typically found in tokamaks.
- Flow damping affects island propagation velocity, which modifies ion polarization term in Rutherford equation.
- Require drift-MHD island theory that takes flow damping into account. [a]

[a] R. Fitzpatrick, and F.L. Waelbroeck, Phys. Plasmas **16**, 072507 (2009).

Finite Trapped Ion Orbit Width

- Width of trapped ion orbit of order

$$\rho_\theta = (B_z/B_\theta)\,\rho.$$

- In conventional tokamak, trapped ion orbit width often comparable with island width.
- Kinetic analysis required to take finite orbit widths into account. [a]

[a] A. Bergmann, E. Poli, and A.G. Peeters, Phys. Plasmas **12**, 072501 (2005).

Magnetic Field-Line Curvature

- Magnetic field-line curvature in tokamak plasmas gives rise to particle drifts that are three-dimensional in nature, and cannot be captured in two-dimensional slab model.
- Three-dimensional island theory required to take curvature drifts into account. [a]

[a] M. Kotschenreuther, R.D. Hazeltine, and P.J. Morrison, Phys. Fluids **28**, 294 (1985).

Drift-Wave Turbulence

- Perpendicular transport that determines island profiles actually due to drift-wave turbulence.
- Radial extent of drift-wave eddies of order ρ. Hence, eddies can easily be comparable in width to island.
- Island theory in which island immersed in bath of drift-wave turbulence required when eddy width comparable with island width. Turbulence affects island by modifying island profiles. Island profiles affect drift-wave stability, and hence turbulence levels. Theory must self-consistently determine effect of turbulence on island, and effect of island on turbulence.[a]

[a]F. Militello, F.L. Waelbroeck, R. Fitzpatrick, and W. Horton, Plasma Phys. Control. Fusion **51**, 015015 (2009).