Multi-Harmonic Rutherford Island Theory

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Tearing Mode

- Consider Harris-type current sheet, running in y-direction, of thickness a in x-direction. Let system be independent of z.
- Suppose that sheet is subject to tearing mode perturbation of wavenumber k_0 in y-direction.
- Mode reconnects magnetic flux at center of sheet to produce magnetic island chain of width W (in x-direction) and wavelength $2\pi/k_0$ (in y-direction).

Rutherford Theory

- As soon as island width exceeds very thin linear layer width (which is of order $S^{-2/5}\,a$, where $S \gg 1$ is Lundquist number), tearing mode enters nonlinear regime.
- According to Rutherford theory,[1] if tearing mode in constant-ψ regime[2] then island width grows algebraically on resistive diffusion timescale, τ_R:

$$0.8227\,\tau_R\,\frac{d}{dt}\left(\frac{W}{a}\right) = \Delta'.$$

- Here, Δ' is tearing stability index (normalized to a).

[1] P.H. Rutherford, Phys. Fluids **16**, 1903 (1973).

[2] H.P. Furth, J. Killeen and M.N. Rosenbluth, Phys. Fluids **6**, 459 (1963).

Critique of Rutherford Theory

- Rutherford theory is nonlinear because (normalized) perturbed current density, J, in vicinity of island is multi-harmonic: i.e.,

$$J(x, y, t) = J_1(x, t) \cos(k_0 y) + \sum_{m=2}^{\infty} J_m(x, t) \cos(m k_0 y).$$

- Overtone harmonics, J_2, J_3, etc., similar in magnitude to fundamental harmonic, J_1.
- But, perturbed (normalized) magnetic flux assumed to consist of fundamental harmonic only,

$$\psi(x, y, t) = \Psi_1(t) \cos(k_0 y).$$

- Is this approximation justified?

Multi-Harmonic Rutherford Theory - I

- Assume that ψ is multi-harmonic: i.e.,

$$\psi(x, y, t) = \Psi_1(t)\,\cos(k_0\,y) + \sum_{m=2}^{\infty} \Psi_m(t)\,\cos(m\,k_0\,y).$$

- Can derive multi-harmonic generalization of Rutherford island width evolution equation:[3]

$$\tau_R\,\frac{W_1}{2}\sum_{m'=1,\infty} I_{m,m'}\,\frac{d\Psi_{m'}}{dt} = \Delta'_m\,\Psi_m - K_m\,\Psi_1. \qquad (1)$$

- Here, $W_1 = 4\,\Psi_1^{1/2}$ is (normalized) island width in single-harmonic limit, and Δ'_m is tearing stability index for mode of wavenumber $m\,k_0$.

[3] R. Fitzpatrick, Phys. Plasmas **29**, 092501 (2022).

Multi-Harmonic Rutherford Theory - II

- Furthermore,

$$I_{m,m'} = 2\int_{\Omega_{\rm min}}^{\infty}\frac{C_m(\Omega)\,C_{m'}(\Omega)}{C_0(\Omega)}\,d\Omega,$$

$$K_m = -16\int_{\Omega_{\rm min}}^{\infty}\langle J_{\rm ni}\rangle\,C_m(\Omega)\,d\Omega,$$

$$C_m(\Omega) = \langle\cos(m\,k_0)\rangle.$$

- Here, Ω is flux-surface label such that $\Omega = 1$ on magnetic separatrix, and $\Omega = \Omega_{\rm min}$ at O-points.
- $J_{\rm ni}$ is non-inductive current density, driven by electron cyclotron waves injected into plasma (standard way of controlling tearing modes in tokamaks).
- $\langle\cdots\rangle$ is flux-surface average operator (i.e., annihilator of $\mathbf{B}\cdot\nabla$).

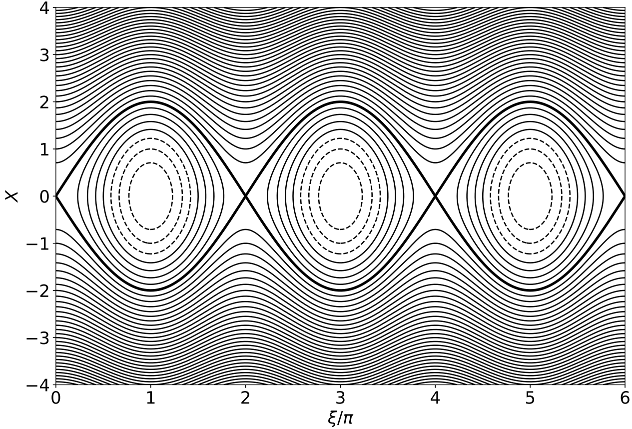
Two-Harmonic Limit - I

- Equation (1) is highly nonlinear because $I_{m,m'}$ and K_m depend on relative magnitudes of Ψ_m.
- In general, equation almost impossible to solve.
- However, equation can be solved in two-harmonic limit in which

$$\Psi(x,y,t) = \Psi_1(t)\,[\cos(k_0\,y) + \epsilon_2\,\cos(2\,k_0\,y)].$$

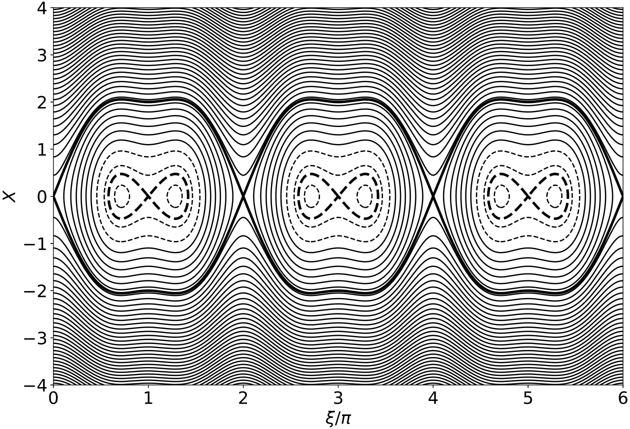
Two-Harmonic Limit - II

- For $-1/4 < \epsilon_2 < 1/4$, magnetic flux-surfaces topologically equivalent to those found in single-harmonic limit.



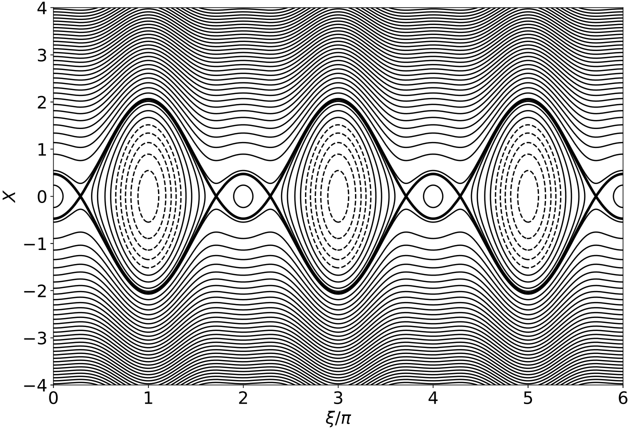
Two-Harmonic Limit - III

- If $\epsilon_2 > 1/4$ then O-points bifurcate.



Two-Harmonic Limit - IV

- If $\epsilon_2 < -1/4$ then X-points bifurcate.



Two-Harmonic Limit - V

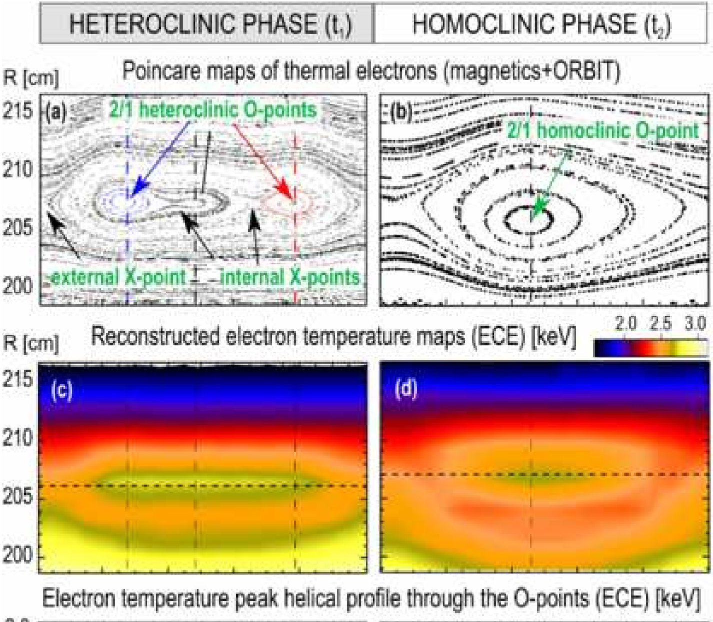
- In absence of non-inductive current, find that $\epsilon_2 \simeq 0$ when $\Delta_1' \ll 1$: i.e., weakly unstable tearing mode is single-harmonic in nature.
- ϵ_2 increases with increasing Δ_1', but increase saturates at $\epsilon_2 = 0.095$, well below level needed to trigger bifurcation of O-points.
- Growth-rate of tearing mode in two-harmonic limit is greater than that in one-harmonic limit, but only by maximum of about 10%.
- Conclusion: single-harmonic approximation works pretty well for tearing modes in absence of non-inductive current.

Two-Harmonic Limit - VI

- Non-inductive current concentrated at island O-points has stabilizing effect on mode. (In practice, this technique used to suppress tearing modes in tokamaks).
- However, non-inductive current drives multi-harmonic content of mode. In fact, ϵ_2 driven past critical value $1/4$, above which O-points bifurcate, before full stabilization achieved.
- This result highly significant, because bifurcation of island O-points, in presence of non-inductive current injected into plasma at island O-points, was recently observed experimentally on DIII-D tokamak.[4]

[4] L. Bardóczi and T.E. Evans, Phys. Rev. Lett. **126**, 085003 (2021).

Experimental Evidence for O-point Bifurcation



Conclusions

- Single-harmonic Rutherford theory is reasonably accurate for isolated magnetic island chains.
- However, if island chain is subject to non-inductive current, injected into plasma in vicinity of O-points, in effort to stabilize chain, then topology of chain can change.
- Implies that single-harmonic theory is not particularly accurate at describing stabilization of magnetic island chains via non-inductive current injection.