Multi-Harmonic Rutherford Island Theory

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Consider Harris-type current sheet, running in $y$-direction, of thickness $a$ in $x$-direction. Let system be independent of $z$.

Suppose that sheet is subject to tearing mode perturbation of wavenumber $k_0$ in $y$-direction.

Mode reconnects magnetic flux at center of sheet to produce magnetic island chain of width $W$ (in $x$-direction) and wavelength $2\pi/k_0$ (in $y$-direction).
As soon as island width exceeds very thin linear layer width (which is of order $S^{-2/5} a$, where $S \gg 1$ is Lundquist number), tearing mode enters nonlinear regime.

According to Rutherford theory,\(^1\) if tearing mode in constant-$\psi$ regime\(^2\) then island width grows algebraically on resistive diffusion timescale, $\tau_R$:

$$0.8227 \tau_R \frac{d}{dt} \left( \frac{W}{a} \right) = \Delta'.$$

Here, $\Delta'$ is tearing stability index (normalized to $a$).

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Rutherford theory is nonlinear because (normalized) perturbed current density, $J$, in vicinity of island is multi-harmonic: i.e.,

$$J(x, y, t) = J_1(x, t) \cos(k_0 y) + \sum_{m=2}^{\infty} J_m(x, t) \cos(m k_0 y).$$

Overtone harmonics, $J_2$, $J_3$, etc., similar in magnitude to fundamental harmonic, $J_1$.

But, perturbed (normalized) magnetic flux assumed to consist of fundamental harmonic only,

$$\psi(x, y, t) = \Psi_1(t) \cos(k_0 y).$$

Is this approximation justified?
Assume that $\psi$ is multi-harmonic: i.e.,

$$\psi(x, y, t) = \Psi_1(t) \cos(k_0 y) + \sum_{m=2}^{\infty} \Psi_m(t) \cos(m k_0 y).$$

Can derive multi-harmonic generalization of Rutherford island width evolution equation:

$$\tau_R \frac{W_1}{2} \sum_{m'=1,\infty} I_{m,m'} \frac{d\Psi_{m'}}{dt} = \Delta'_m \Psi_m - K_m \Psi_1. \quad (1)$$

Here, $W_1 = 4 \Psi_1^{1/2}$ is (normalized) island width in single-harmonic limit, and $\Delta'_m$ is tearing stability index for mode of wavenumber $m k_0$.

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Furthermore,

\[ I_{m,m'} = 2 \int_{\Omega_{\text{min}}}^{\infty} \frac{C_m(\Omega) C_{m'}(\Omega)}{C_0(\Omega)} \, d\Omega, \]

\[ K_m = -16 \int_{\Omega_{\text{min}}}^{\infty} \langle J_{ni} \rangle C_m(\Omega) \, d\Omega, \]

\[ C_m(\Omega) = \langle \cos(m k_0) \rangle. \]

Here, \( \Omega \) is flux-surface label such that \( \Omega = 1 \) on magnetic separatrix, and \( \Omega = \Omega_{\text{min}} \) at O-points.

\( J_{ni} \) is non-inductive current density, driven by electron cyclotron waves injected into plasma (standard way of controlling tearing modes in tokamaks).

\( \langle \cdots \rangle \) is flux-surface average operator (i.e., annihilator of \( B \cdot \nabla \)).
Equation (1) is highly nonlinear because $I_{m,m'}$ and $K_m$ depend on relative magnitudes of $\Psi_m$.

In general, equation almost impossible to solve.

However, equation can be solved in two-harmonic limit in which

$$\Psi(x, y, t) = \Psi_1(t) [\cos(k_0 y) + \epsilon_2 \cos(2 k_0 y)].$$
For $-1/4 < \epsilon_2 < 1/4$, magnetic flux-surfaces topologically equivalent to those found in single-harmonic limit.
If $\epsilon_2 > 1/4$ then O-points bifurcate.
If $\epsilon_2 < -1/4$ then X-points bifurcate.
In absence of non-inductive current, find that $\epsilon_2 \approx 0$ when $\Delta'_1 \ll 1$: i.e., weakly unstable tearing mode is single-harmonic in nature.

$\epsilon_2$ increases with increasing $\Delta'_1$, but increase saturates at $\epsilon_2 = 0.095$, well below level needed to trigger bifurcation of $O$-points.

Growth-rate of tearing mode in two-harmonic limit is greater than that in one-harmonic limit, but only by maximum of about 10%.

**Conclusion**: single-harmonic approximation works pretty well for tearing modes in absence of non-inductive current.
Non-inductive current concentrated at island O-points has stabilizing effect on mode. (In practice, this technique used to suppress tearing modes in tokamaks).

However, non-inductive current drives multi-harmonic content of mode. In fact, $\varepsilon_2$ driven past critical value $1/4$, above which O-points bifurcate, before full stabilization achieved.

This result highly significant, because bifurcation of island O-points, in presence of non-inductive current injected into plasma at island O-points, was recently observed experimentally on DIII-D tokamak.\(^4\)

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Experimental Evidence for O-point Bifurcation
Conclusions

- Single-harmonic Rutherford theory is reasonably accurate for isolated magnetic island chains.

- However, if island chain is subject to non-inductive current, injected into plasma in vicinity of O-points, in effort to stabilize chain, then topology of chain can change.

- Implies that single-harmonic theory is not particularly accurate at describing stabilization of magnetic island chains via non-inductive current injection.