ARTICLE COMMENTARY | OCTOBER 02 2025

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Phys. Plasmas 32, 104701 (2025)
https://doi.org/10.1063/5.0296028

Physics of Plasmas COMMENT pubs.aip.org/aip/pop

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Cite as: Phys. Plasmas **32**, 104701 (2025); doi: 10.1063/5.0296028
Submitted: 12 August 2025 · Accepted: 20 September 2025 ·
Published Online: 2 October 2025

View Online Export Citation CrossMark

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ABSTRACT

A recent paper by Chang suggests replacing the conventional expression for the Coulomb logarithm, that appears in charged particle scattering theory in weakly coupled plasmas, with a new and improved expression. In this comment, we demonstrate that the new expression is completely equivalent to the old one, and that any improvements are illusory.

I. CHARGE PARTICLE SCATTERING THEORY

Let us start by reviewing the usual treatment of the scattering of charged particles in a fully ionized weakly coupled plasma.

Suppose that the plasma consists of two charge species, labeled s and s', with respective masses m_s and $m_{s'}$, electrical charges e_s and $e_{s'}$, number densities n_s and $n_{s'}$, and temperatures T_s and $T_{s'}$. A collision between a particle of species s and a particle of species s' is such that[1]

$$\cot\left(\frac{\chi}{2}\right) = \frac{\mu_{ss'}\, u_{ss'}^2\, b}{\kappa_{ss'}}, \tag{1}$$

where χ is the scattering angle, $\mu_{ss'} = (1/m_s + 1/m_{s'})^{-1}$ the reduced mass, b the impact parameter, $u_{ss'}$ the relative speed of the two particles before and after the collision, and

$$\kappa_{ss'} = \frac{e_s\, e_{s'}}{4\pi\,\epsilon_0}. \tag{2}$$

The Coulomb logarithm is defined as[1]

$$\Lambda = \int \frac{d(\chi/2)}{\sin(\chi/2)}, \tag{3}$$

where the integral is over all scattering angle. As is well known, this integral does not diverge at large scattering angles (i.e., $\chi \to \pi$), but does diverge at small scattering angles (i.e., $\chi \to 0$). The divergence is indicative of the fact that the dominant contribution to the integral comes from small-angle scattering events. If we make the small-angle approximation, then

$$\chi \simeq \frac{2\,\kappa_{ss'}}{\mu_{ss'}\, u_{ss'}^2\, b} \tag{4}$$

and

$$\Lambda \simeq \int \frac{d\chi}{\chi} = \ln\left(\frac{\chi_{\max}}{\chi_{\min}}\right) = \ln\left(\frac{b_{\max}}{b_{\min}}\right). \tag{5}$$

The usual choice for $b_{\min}$ is the impact parameter for 90° scattering,

$$b_{\min} = \frac{\kappa_{ss'}}{\mu_{ss'}\, u_{ss'}^2}. \tag{6}$$

As explained in Ref. 2, and references therein, this result can be obtained by asymptotically matching the small-angle scattering solution to a solution that treats large-angle scattering correctly (and, therefore, does not diverge as $b \to 0$). The usual choice for $b_{\max}$ is the Debye length, λ_D. This result comes from the assumption that the true scattering potential energy between a particle of species s and a particle of species s' is

$$U_{ss'}(r) = \frac{\kappa_{ss'}}{r}\, \exp\left(-\frac{\sqrt{2}\,r}{\lambda_D}\right), \tag{7}$$

where the correction is due to Debye shielding on length scales longer than λ_D. The whole treatment of Coulomb collisions as primarily small-angle binary scattering event depends on the ordering

$$b_{\min} \ll b_{\max}, \tag{8}$$

which implies that $\Lambda \gg 1$. Finally, if both species s and s' have Maxwellian velocity distributions, with no relative drift, then[1]

$$\frac{dT_s}{dt} = -\frac{T_s - T_{s'}}{\tau_{\rm eq}}, \tag{9}$$

where

$$\tau_{\rm eq} = \frac{3}{32\,\pi^{1/2}}\,\frac{m_s\, m_{s'}\,(v_{ts}^2 + v_{ts'}^2)^{3/2}}{\kappa_{ss'}^2\, n_{s'}\,\Lambda}, \tag{10}$$

and $v_{ts} = \sqrt{2\,T_s/m_s}$ and $v_{ts'} = \sqrt{2\,T_{s'}/m_{s'}}$.

II. MAIN RESULT OF PAPER

If we look at Dr. Chang's expression (108) for the temperature equilibration time, then, unpacking all of the previous definitions in the paper, we find that

$$\tau_{\rm eq} = \frac{3}{32\,\pi^{1/2}}\,\frac{m_s\, m_{s'}\,(v_{ts}^2 + v_{ts'}^2)^{3/2}}{\kappa_{ss'}^2\, n_{s'}\,\lambda}, \tag{11}$$

where

$$\lambda = \frac{1}{2}\,\Gamma(0,x) = \frac{1}{2}\,E_1(x). \tag{12}$$

Here, $\Gamma(a,x)$ is an incomplete gamma function, $E_1(x)$ is an exponential integral,[3]

$$x = \frac{m_s^2\,\delta_{\min}^2}{8\,\mu_{ss'}\,T_{ss'}} \tag{13}$$

and

$$T_{ss'} = \mu_{ss'}\left(\frac{T_s}{m_s} + \frac{T_{s'}}{m_{s'}}\right), \tag{14}$$

where $\delta = |\mathbf{v}_s - \mathbf{v}_{s'}|$. Thus, as is clear from a comparison of Eqs. (10) and (11), the parameter λ, specified in Eq. (12), plays the role of the conventional Coulomb logarithm Λ, specified in Eq. (5), in Dr. Chang's analysis. In fact, this is the main result of the paper.

III. RELATION TO CONVENTIONAL THEORY

According to Eqs. (A1) and (A2) in the paper,

$$m_s\,\delta = 2\,\mu_{ss'}\,u_{ss'}\,\sin\left(\frac{\chi}{2}\right) \simeq \mu_{ss'}\,u_{ss'}\,\chi, \tag{15}$$

where we have made the small-angle approximation (because we are trying to calculate $\delta_{\min}$, which is much less than $u_{ss'}$). Combining with Eqs. (4) and (6), we get

$$\delta_{\min} = \frac{2\,\kappa_{ss'}}{m_s\,u_{ss'}\,b_{\max}} = 2\,\frac{b_{\min}}{b_{\max}}\,\frac{\mu_{ss'}}{m_s}\,u_{ss'}. \tag{16}$$

Thus,

$$x = \frac{\mu_{ss'}\,u_{ss'}^2}{2\,T_{ss'}}\left(\frac{b_{\min}}{b_{\max}}\right)^2. \tag{17}$$

However,

$$\langle u_{ss'}^2\rangle = \frac{3\,T_{ss'}}{\mu_{ss'}}, \tag{18}$$

so replacing x by its average value, we get

$$x = \frac{3}{2}\left(\frac{b_{\min}}{b_{\max}}\right)^2 = \frac{3}{2}\,\exp(-2\,\Lambda). \tag{19}$$

Now, the whole idea of treating collisions as primarily binary events depends on the ordering $b_{\max} \gg b_{\min}$. Hence, we conclude that x is extremely small. (In fact, in a typical fusion plasma with $\Lambda \simeq 15$ we find that $x \sim 1 \times 10^{-13}$.) Thus, it is an excellent approximation to use the small-argument expansion of the exponential integral,[3]

$$E_1(x) = -\gamma - \ln x + \mathcal{O}(x), \tag{20}$$

where γ is Euler's constant, to give

$$\lambda = \frac{1}{2}\left(-\gamma - \ln\left[\frac{3}{2}\left(\frac{b_{\min}}{b_{\max}}\right)^2\right]\right) = -0.49 + \ln\left(\frac{b_{\max}}{b_{\min}}\right) = \ln\left(\frac{0.61\,b_{\max}}{b_{\min}}\right). \tag{21}$$

In conclusion,

$$\lambda = \Lambda - 0.49 + \mathcal{O}(e^{-2\Lambda}). \tag{22}$$

IV. CONCLUSION

As is clear from the previous equation, Dr. Chang's replacement for the Coulomb logarithm only differs from the standard result by a constant whose magnitude is less than unity. However, conventional Coulomb scattering theory in a weakly coupled plasma depends on the ordering $\Lambda \gg 1$. In fact, the theory is essentially the first term in an expansion in $1/\Lambda$, which means that there is an intrinsic uncertainty in Λ that is of order unity. It, therefore, follows that Dr. Chang's analysis is not an improvement on the conventional analysis. In fact, it is just an unnecessarily complicated way of getting essentially the same result.

ACKNOWLEDGMENTS

This research was funded by the U.S. Department of Energy, Office of Science, Office of Fusion Energy Sciences under Contract No. DE-SC0021156.

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