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Interaction of a magnetic island chain in a tokamak plasma with a resonant magnetic perturbation of rapidly oscillating phase

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An investigation is made into the interaction of a magnetic island chain, embedded in a tokamak plasma, with an externally generated magnetic perturbation of the same helicity whose helical phase is rapidly oscillating. The analysis is similar in form to the classic analysis used by Kapitza [Sov. Phys. JETP **21**, 588 (1951)] to examine the angular motion of a rigid pendulum whose pivot point undergoes rapid vertical oscillations. The phase oscillations are found to modify the existing terms, and also to give rise to new terms, in the equations governing the secular evolution of the island chain's radial width and helical phase. An examination of the properties of the new secular evolution equation reveals that it is possible to phase-lock an island chain to an external magnetic perturbation with an oscillating helical phase in a *stabilizing* phase relation provided that the amplitude, ϵ, of the phase oscillations (in radians) is such that $|J_0(\epsilon)| \ll 1$, and the mean angular frequency of the perturbation closely matches the natural angular frequency of the island chain. *Published by AIP Publishing*. https://doi.org/10.1063/1.5000253

I. INTRODUCTION

A tokamak is a device that is designed to trap a thermonuclear plasma on a set of axisymmetric, toroidally nested, magnetic flux-surfaces.[1] Heat and particles are able to flow around the flux-surfaces relatively rapidly due to the free streaming of charged particles along magnetic field-lines. On the other hand, heat and particles can only diffuse across the flux-surfaces relatively slowly, assuming that the magnetic field-strength is large enough to render the particle gyroradii much smaller than the device's minor radius.[2]

Tokamak plasmas are subject to a number of macroscopic instabilities that limit their effectiveness.[3] So-called *tearing modes* are comparatively slowly growing instabilities[4] that saturate at relatively low amplitudes,[5–9] in the process reconnecting magnetic flux-surfaces to form helical structures known as *magnetic island chains*. Magnetic island chains are radially localized structures centered on so-called *rational magnetic flux-surfaces* which satisfy the resonance criterion $\mathbf{k} \cdot \mathbf{B} = 0$, where $\mathbf{k}$ is the wave-number of the instability and $\mathbf{B}$ the equilibrium magnetic field. Magnetic islands degrade plasma confinement because they enable heat and particles to flow very rapidly along field-lines from their inner to their outer radii, implying an almost complete loss of confinement in the region lying between these radii.[10]

In many previous tokamak experiments, an externally generated, rotating, magnetic perturbation, resonant at some rational surface that lies within the plasma, was employed in an attempt to reduce the amplitude of a saturated magnetic island chain that was localized at the same rational surface (see, for instance, Refs. 11, 12 and 13). A major obstacle to this method of tearing mode suppression is the existence of the so-called *phase instability*, which causes the helical phase of a magnetic island chain that has a stabilizing phase relation to an external magnetic perturbation to spontaneously flip, such that the chain has a destabilizing phase relation, on a comparatively short time scale.[14] Rotating magnetic perturbations can also be used to modify the flow profile in a tokamak because they exert an electromagnetic torque on the plasma in the immediate vicinity of a saturated magnetic island chain of the same helicity (see, for instance, Ref. 15). Unfortunately, the existence of the phase instability ensures that the island chain always phase-locks to the rotating perturbation in such a manner that the chain is destabilized, which has a deleterious effect on plasma energy and particle confinement.

The electromagnetic torque exerted by a resonant external magnetic perturbation on the plasma in the vicinity of a magnetic island chain of the same helicity varies as the sine of the helical phase difference between the chain and the perturbation.[16] This torque modifies the plasma flow in the vicinity of the island chain, which, in turn, changes the chain's helical phase because the chain is constrained to corotate with the plasma at the rational surface.[16] Now, the gravitational torque exerted on a rigid pendulum depends on the sine of the angle subtended between the pendulum and the downward vertical. It follows that there is an analogy between the phase evolution of a magnetic island chain interacting with a resonant external magnetic perturbation and the angular motion of a rigid pendulum. In the case of a rigid pendulum, there are two equilibrium states. The first, in which the pendulum's center of mass is directly below the pivot point, is dynamically stable. The second, in which the center of mass is directly above the pivot point, is dynamically unstable. Likewise, in the case of a magnetic island chain interacting with a resonant external magnetic perturbation of the same helicity (in the absence of plasma flow), there are two equilibrium states.[16] The first, in which the helical phase difference between the island chain and the external perturbation is zero, is dynamically stable. The second, in which the helical phase difference between the island chain and the external perturbation is π radians, is dynamically unstable. Unfortunately, the dynamically stable equilibrium state is such that the island chain is maximally destabilized by the external perturbation and vice versa.[16]

It is well known that if the pivot point of a rigid pendulum is made to execute a small-amplitude, vertical oscillation of sufficiently high frequency, then the equilibrium state in which the pendulum's center of mass lies directly above the pivot point can be rendered dynamically stable, whilst the equilibrium state in which the center of mass lies directly below the pivot point is rendered dynamically unstable.[17,18] The analogy that exists between the phase evolution of a magnetic island chain interacting with a resonant external magnetic perturbation of the same helicity and the angular motion of a rigid pendulum leads us to speculate that if the helical phase of the external perturbation were subject to a small amplitude, high frequency oscillation, then it might be possible to induce the island chain to lock to the perturbation in the maximally stabilizing phase relation. Such a phenomenon, if it existed, would greatly facilitate the magnetic control of tearing modes in tokamak plasmas and would also prevent any confinement degradation associated with the modification of the plasma flow profile by rotating external magnetic perturbations. The aim of this paper is to explore the aforementioned possibilities.

II. PRELIMINARY ANALYSIS

A. Plasma equilibrium

Consider a large aspect-ratio, low-β [i.e., $\beta \sim \mathcal{O}(a/R_0)^2$], tokamak plasma whose magnetic flux surfaces map out (almost) concentric circles in the poloidal plane. Such a plasma is well approximated as a periodic cylinder. Suppose that the minor radius of the plasma is a. Standard cylindrical coordinates (r, θ, z) are adopted. The system is assumed to be periodic in the z-direction, with periodicity length $2\pi R_0$, where $R_0 \gg a$ is the simulated plasma major radius. It is convenient to define the simulated toroidal angle $\phi = z/R_0$.

The equilibrium magnetic field is written as $\mathbf{B} = [0, B_\theta(r), B_\phi]$, where $B_\theta \geq 0$ and $B_\phi > 0$. The associated equilibrium plasma current density takes the form $\mathbf{j} = [0, 0, j_\phi(r)]$, where

$$\mu_0 j_\phi(r) = \frac{1}{r}\frac{d(r B_\theta)}{dr}, \tag{1}$$

and $j_\phi \geq 0$. Finally, the *safety factor*

$$q(r) = \frac{r B_\theta}{R_0 B_\phi} \tag{2}$$

parameterizes the helical pitch of equilibrium magnetic field-lines. In a conventional tokamak plasma, $q(r)$ is positive, of order unity, and a monotonically increasing function of r.

B. Plasma response to external helical magnetic perturbation

Consider the response of the plasma to an externally generated, helical magnetic perturbation. Suppose that the magnetic perturbation has $m > 0$ periods in the poloidal direction and $n > 0$ periods in the toroidal direction. It is convenient to express the perturbed magnetic field and the perturbed plasma current density in terms of a magnetic flux-function, $\psi(r, \theta, \phi, t)$. Thus

$$\delta\mathbf{B} = \nabla\psi \times \mathbf{e}_z, \tag{3}$$

$$\mu_0\, \delta\mathbf{j} = -\nabla^2\psi\, \mathbf{e}_z, \tag{4}$$

where

$$\psi(r, \theta, \phi, t) = \hat{\psi}(r, t)\, \exp\left[\,\mathrm{i}\,(m\,\theta - n\,\phi)\right]. \tag{5}$$

This representation is valid, provided that[16]

$$\frac{m}{n} \gg \frac{a}{R_0}. \tag{6}$$

As is well known, the response of the plasma to the external magnetic perturbation is governed by the equations of perturbed, marginally stable (i.e., $\partial/\partial t \equiv 0$), ideal magnetohydrodynamics (MHD) everywhere in the plasma, apart from a relatively narrow (in r) region in the vicinity of the rational surface, minor radius r_s, where $q(r_s) = m/n$.[4,16]

It is convenient to parameterize the external magnetic perturbation in terms of the so-called *vacuum flux*, $\Psi_v(t) = |\Psi_v|\, \mathrm{e}^{-\mathrm{i}\varphi_v}$, which is defined to be the value of $\hat{\psi}(r, t)$ at radius r_s in the presence of the external perturbation, but in the absence of the plasma. Here, φ_v is the helical phase of the external perturbation. Likewise, the response of the plasma in the vicinity of the rational surface to the external perturbation is parameterized in terms of the so-called *reconnected flux*, $\Psi_s(t) = |\Psi_s|\, \mathrm{e}^{-\mathrm{i}\varphi_s}$, which is the actual value of $\hat{\psi}(r, t)$ at radius r_s. Here, φ_s is the helical phase of the reconnected flux.

The linear stability of the m, n tearing mode is governed by the *tearing stability index*[4]

$$\Delta' = \left[\frac{d\ln\hat{\psi}}{d\ln r}\right]_{r_s-}^{r_s+}, \tag{7}$$

where $\hat{\psi}(r)$ is a solution of the marginally stable, ideal-MHD equations for the case of an m, n helical perturbation that satisfies physical boundary conditions at $r = 0$ and $r = a$ (in the absence of the externally generated perturbation). According to standard resistive-MHD theory,[4,5] if $\Delta' > 0$, then the m, n tearing mode spontaneously reconnects magnetic flux at the rational surface to form a helical magnetic island chain.

C. Time evolution of island width

The time evolution of the radial width of the m, n magnetic island chain is governed by the *Rutherford equation*[5]

$$I_1\, \tau_R\, \frac{d}{dt}\left(\frac{W}{r_s}\right) = \Delta'(W) + 2\, m \left(\frac{W_v}{W}\right)^2 \cos\left(\varphi_s - \varphi_v\right), \tag{8}$$

where $I_1 = 0.8227$. Here

$$W = 4\left(\frac{R_0\, q_s\, |\Psi_s|}{s_s\, B_\phi}\right)^{1/2} \tag{9}$$

is the full (radial) width of the chain that forms at the rational surface, $q_s = q(r_s)$, and $s_s = (d\ln q/d\ln r)_{r=r_s}$ is the local magnetic shear. The nonlinear dependence of Δ' on the island width is specified in Refs. 7–9. The quantity

$$W_v = 4\left(\frac{R_0\, q_s\, |\Psi_v|}{s_s\, B_\phi}\right)^{1/2} \tag{10}$$

is termed the *vacuum island width*. It is assumed that $W/a \ll 1$ and $W_v/a \ll 1$. Finally

$$\tau_R = \mu_0\, r_s^2\, \sigma(r_s) \tag{11}$$

is the *resistive diffusion time* at the rational surface, where $\sigma(r)$ is the equilibrium plasma electrical conductivity profile.

D. Electromagnetic torque

It is easily demonstrated that zero net electromagnetic torque can be exerted on magnetic flux surfaces located in a region of the plasma that is governed by the equations of marginally stable, ideal-MHD.[16] Thus, any electromagnetic torque exerted on the plasma by the external perturbation develops in the immediate vicinity of the rational surface, where ideal-MHD breaks down. In fact, the net toroidal electromagnetic torque exerted in the vicinity of the rational surface by the external perturbation takes the form[16]

$$T_{\phi\rm EM} = \frac{4\pi^2\, n\, m\, R_0}{\mu_0}\, |\Psi_s|\, |\Psi_v|\, \sin(\varphi_s - \varphi_v). \tag{12}$$

E. Plasma toroidal equation of angular motion

The plasma's toroidal equation of angular motion is written as[16]

$$4\pi^2 R_0^3\left[\rho r\frac{\partial \Delta\Omega_\phi}{\partial t} - \frac{\partial}{\partial r}\left(\mu r\frac{\partial\Delta\Omega_\phi}{\partial r}\right)\right] = T_{\phi\rm EM}\,\delta(r-r_s). \tag{13}$$

Here, $\Delta\Omega_\phi(r,t)$, $\rho(r)$, and $\mu(r)$ are the plasma toroidal angular velocity-shift (due to the electromagnetic torque), mass density, and (perpendicular) viscosity profiles, respectively. The physical boundary conditions are[16]

$$\frac{\partial\Delta\Omega_\phi(0,t)}{\partial r} = \Delta\Omega_\phi(a,t) = 0. \tag{14}$$

F. Time evolution of island phase

The time evolution of the helical phase of the m, n magnetic island chain is governed by the so-called *no-slip constraint*[16]

$$\frac{d\varphi_s}{dt} = -n\left[\Omega_\phi(r_s) + \Delta\Omega_\phi(r_s,t)\right], \tag{15}$$

according to which the chain is forced to co-rotate with the plasma at the rational surface. Here, $\Omega_\phi(r)$ is the unperturbed (by the external perturbation) plasma toroidal angular velocity profile. The no-slip constraint holds provided that the island width exceeds the linear layer width.[19] Note that we are neglecting poloidal plasma rotation in this study, which is reasonable because in tokamak plasmas such rotation is strongly constrained by poloidal flow damping.[16,20] We are also neglecting two-fluid effects; however, such effects can easily be incorporated into the analysis because, to lowest order, they merely introduce a constant diamagnetic offset between the island chain's velocity (in the diamagnetic direction) and that of the guiding center fluid at the rational surface.[21–23]

G. Normalization

Suppose, for the sake of simplicity, that both the mass density and (perpendicular) viscosity are spatially uniform across the plasma, and also constant in time. It is helpful to define the *momentum confinement time*

$$\tau_M = \frac{\rho\, a^2}{\mu}, \tag{16}$$

the *hydromagnetic time*

$$\tau_H = \left(\frac{R_0}{n\, s_s}\right)\left(\frac{\mu_0\,\rho}{B_\phi^2}\right)^{1/2}, \tag{17}$$

the *island width evolution time*

$$\tau_W = \frac{I_1}{2\,m}\,\frac{W_v}{r_s}\,\tau_R, \tag{18}$$

the *island phase evolution time*

$$\tau_\varphi = \frac{2^4}{\sqrt{m}}\,\frac{q_s}{\epsilon_a}\left(\frac{a}{W_v}\right)^2\tau_H, \tag{19}$$

where $\epsilon_a = a/R_0$, and the *natural frequency*

$$\omega_s = -n\Omega_\phi(r_s). \tag{20}$$

The latter quantity is the angular frequency of the m, n tearing mode in the absence of the externally generated perturbation.

Let $\hat{r} = r/a$, $\hat{r}_s = r_s/a$, $\hat{t} = t/\tau_M$, $\hat{\tau}_W = \tau_W/\tau_M$, $\hat{\tau}_\varphi = \tau_\varphi/\tau_M$, $\hat{W} = W/W_v$, $\hat{\omega}_s = \omega_s\,\tau_M$, and $\Omega(\hat{r},\hat{t}) = -n\,\Delta\Omega_\phi\,\tau_M$. It follows that

$$\hat{\tau}_W\,\frac{d\hat{W}}{d\hat{t}} = \frac{\Delta'}{2\,m} + \hat{W}^{-2}\cos(\varphi_s - \varphi_v), \tag{21}$$

$$\hat{\tau}_\varphi^2\left[\frac{\partial\Omega}{\partial\hat{t}} - \frac{1}{\hat{r}}\frac{\partial}{\partial\hat{r}}\left(\hat{r}\,\frac{\partial\Omega}{\partial\hat{r}}\right)\right] = -\hat{W}^2\sin(\varphi_s - \varphi_v)\,\frac{\delta(\hat{r}-\hat{r}_s)}{\hat{r}}, \tag{22}$$

$$\frac{\partial\Omega(0,\hat{t})}{\partial\hat{r}} = \Omega(1,\hat{t}) = 0, \tag{23}$$

$$\frac{d\varphi_s}{d\hat{t}} = \hat{\omega}_s + \Omega(\hat{r}_s,\hat{t}). \tag{24}$$

H. Derivation of phase evolution equations

Let

$$u_k(\hat{r}) = \frac{\sqrt{2}\,J_0(j_{0,k}\,\hat{r})}{J_1(j_{0,k})}, \tag{25}$$

where k is a positive integer and $j_{0,k}$ denotes the kth zero of the J_0 Bessel function. It is easily demonstrated that[24]

$$\int_0^1 \hat{r}\,u_k(\hat{r})\,u_{k'}(\hat{r})\,d\hat{r} = \delta_{kk'}, \tag{26}$$

and[25]

$$\frac{\delta(\hat{r}-\hat{r}_s)}{\hat{r}} = \sum_{k=1,\infty} u_k(\hat{r}_s)\,u_k(\hat{r}). \tag{27}$$

Let us write[26]

$$\Omega(\hat{r},\hat{t}) = \sum_{k=1,\infty} h_k(\hat{t})\,\frac{u_k(\hat{r})}{u_k(\hat{r}_s)}, \tag{28}$$

which automatically satisfies the spatial boundary conditions (23). Substitution into Eq. (22) yields the m, n island *phase evolution equations*[26]

$$\hat{\tau}_\varphi^2\left(\frac{dh_k}{d\hat{t}} + j_{0,k}^2\,h_k\right) = -[u_k(\hat{r}_s)]^2\,\hat{W}^2\,\sin(\varphi_s-\varphi_v), \tag{29}$$

$$\frac{d\varphi_s}{d\hat{t}} = \hat{\omega}_s + \sum_{k=1,\infty} h_k. \tag{30}$$

III. RESPONSE TO MAGNETIC PERTURBATION WITH RAPIDLY OSCILLATING PHASE

A. Introduction

Suppose that

$$\varphi_v(\hat{t}) = \hat{\omega}_v\,\hat{t} + \epsilon\,\cos(\hat{\omega}_f\,\hat{t}), \tag{31}$$

where $\hat{\omega}_v = \omega_v\tau_M$ and $\hat{\omega}_f = \omega_f\tau_M$. This implies that, on average, the external magnetic perturbation is rotating steadily at the angular velocity ω_v, but that its helical phase is also oscillating sinusoidally about its mean value with amplitude ϵ and frequency ω_f. It is easily demonstrated that[27]

$$\begin{aligned}\cos(\varphi_s-\varphi_v) &= \cos\left[\varphi-\epsilon\,\cos(\hat{\omega}_f\,\hat{t})\right] = \cos\varphi\,\cos\left[\epsilon\,\cos(\hat{\omega}_f\,\hat{t})\right] + \sin\varphi\,\sin\left[\epsilon\,\cos(\hat{\omega}_f\,\hat{t})\right]\\ &= J_0(\epsilon)\,\cos\varphi + 2\sum_{p=1,\infty} J_p(\epsilon)\,\cos\left(\varphi - p\,\frac{\pi}{2}\right)\cos(p\,\hat{\omega}_f\,\hat{t}),\end{aligned} \tag{32}$$

$$\begin{aligned}\sin(\varphi_s-\varphi_v) &= \sin\left[\varphi-\epsilon\,\cos(\hat{\omega}_f\,\hat{t})\right] = \sin\varphi\,\cos\left[\epsilon\,\cos(\hat{\omega}_f\,\hat{t})\right] - \cos\varphi\,\sin\left[\epsilon\,\cos(\hat{\omega}_f\,\hat{t})\right]\\ &= J_0(\epsilon)\,\sin\varphi + 2\sum_{p=1,\infty} J_p(\epsilon)\,\sin\left(\varphi - p\,\frac{\pi}{2}\right)\cos(p\,\hat{\omega}_f\,\hat{t}),\end{aligned} \tag{33}$$

where the $J_p(z)$ are Bessel functions and

$$\varphi = \varphi_s - \hat{\omega}_v\,\hat{t}. \tag{34}$$

The Rutherford equation, (21), and the phase evolution equations, (29) and (30), yield

$$\hat{\tau}_W\,\frac{d\hat{W}}{d\hat{t}} = \frac{\Delta'}{2m} + \hat{W}^{-2} J_0(\epsilon)\cos\varphi + 2\,\hat{W}^{-2}\times\sum_{p=1,\infty} J_p(\epsilon)\cos\left(\varphi - p\frac{\pi}{2}\right)\cos(p\hat{\omega}_f\hat{t}), \tag{35}$$

$$\hat{\tau}_\varphi^2\left(\frac{dh_k}{d\hat{t}} + j_{0,k}^2\,h_k\right) = -[u_k(\hat{r}_s)]^2\,\hat{W}^2 J_0(\epsilon)\,\sin\varphi - 2\,[u_k(\hat{r}_s)]^2\,\hat{W}^2\times\sum_{p=1,\infty} J_p(\epsilon)\,\sin\left(\varphi - p\,\frac{\pi}{2}\right)\cos(p\,\hat{\omega}_f\,\hat{t}), \tag{36}$$

$$\frac{d\varphi}{d\hat{t}} = \hat{\omega}_o + \sum_{k=1,\infty} h_k, \tag{37}$$

where

$$\hat{\omega}_o = (\omega_s - \omega_v)\,\tau_M. \tag{38}$$

In the following, we shall refer to $\omega_o = \omega_s - \omega_v$, which is the difference between the natural angular frequency of the m, n tearing mode and the mean angular frequency of the externally applied m, n magnetic perturbation, as the *offset-frequency*.

B. Two time-scale analysis

Following the work of Kapitza,[18,28] we can write

$$\hat{W} = \bar{W} + \tilde{W}, \tag{39}$$

$$\varphi = \bar{\varphi} + \tilde{\varphi}, \tag{40}$$

$$h_k = \bar{h}_k + \tilde{h}_k. \tag{41}$$

Here, $\tilde{W}$, $\tilde{\varphi}$, and $\tilde{h}_k$ are periodic functions of t with period $2\pi/\omega_f$, whereas $\bar{W}$, $\bar{\varphi}$, and $\bar{h}_k$ vary on a much longer time scale. Let $\langle\cdots\rangle = (\hat{\omega}_f/2\pi)\int_0^{2\pi/\hat{\omega}_f}(\cdots)\,d\hat{t}$ denote an average over the phase-oscillation time. It follows that $\langle\tilde{W}\rangle = \langle\tilde{\phi}\rangle = \langle\tilde{h}_k\rangle = 0$. It is assumed that $|\tilde{W}| \ll |\bar{W}|$, $|\tilde{\varphi}| \ll |\bar{\varphi}|$, and $|\tilde{h}_k| \ll |\bar{h}_k|$.

Equations (35)–(37) yield

$$\begin{aligned}\hat{\tau}_W\left(\frac{d\bar{W}}{d\hat{t}} + \frac{d\tilde{W}}{d\hat{t}}\right) &\simeq \frac{\Delta'}{2\,m} + \bar{W}^{-2}\left(1 - 2\,\frac{\tilde{W}}{\bar{W}}\right)J_0(\epsilon)\,(\cos\bar{\varphi} - \sin\bar{\varphi}\;\tilde{\varphi})\\ &+ 2\,\bar{W}^{-2}\left(1 - 2\,\frac{\tilde{W}}{\bar{W}}\right)\sum_{p=1,\infty} J_p(\epsilon)\times\left[\cos\left(\bar{\varphi} - p\,\frac{\pi}{2}\right)\right.\\ &\left.-\sin\left(\bar{\varphi} - p\,\frac{\pi}{2}\right)\tilde{\varphi}\right]\cos(p\,\hat{\omega}_f\,\hat{t}),\end{aligned} \tag{42}$$

$$\begin{aligned}\hat{\tau}_\varphi^2\left(\frac{d\bar{h}_k}{d\hat{t}} + \frac{d\tilde{h}_k}{d\hat{t}} + j_{0,k}^2\,\bar{h}_k + j_{0,k}^2\,\tilde{h}_k\right) &\simeq -[u_k(\hat{r}_s)]^2\,\bar{W}^2\left(1 + 2\,\frac{\tilde{W}}{\bar{W}}\right)J_0(\epsilon)\,(\sin\bar{\varphi} + \cos\bar{\varphi}\;\tilde{\varphi}) - 2\,[u_k(\hat{r}_s)]^2\,\bar{W}^2\left(1 + 2\,\frac{\tilde{W}}{\bar{W}}\right)\\ &\times\sum_{p=1,\infty} J_p(\epsilon)\times\left[\sin\left(\bar{\varphi} - p\,\frac{\pi}{2}\right) + \cos\left(\bar{\varphi} - p\,\frac{\pi}{2}\right)\tilde{\varphi}\right]\cos(p\,\hat{\omega}_f\,\hat{t}),\end{aligned} \tag{43}$$

$$\frac{d\bar{\varphi}}{d\hat{t}} + \frac{d\tilde{\varphi}}{d\hat{t}} = \hat{\omega}_o + \sum_{k=1,\infty}\left(\bar{h}_k + \tilde{h}_k\right). \tag{44}$$

The oscillating components of the previous three equations (which can be equated separately[28]) give

$$\hat{\tau}_W\,\frac{d\tilde{W}}{d\hat{t}} \simeq -2\bar{W}^{-3}J_0(\epsilon)\cos\bar{\varphi}\,\tilde{W} - \bar{W}^{-2}J_0(\epsilon)\sin\bar{\varphi}\,\tilde{\varphi} + 2\bar{W}^{-2}\sum_{p=1,\infty}J_p(\epsilon)\cos\left(\bar{\varphi}-p\,\frac{\pi}{2}\right)\cos(p\,\hat{\omega}_f\,\hat{t}), \tag{45}$$

$$\hat{\tau}_\varphi^2\left(\frac{d\tilde{h}_k}{d\hat{t}} + j_{0,k}^2\,\tilde{h}_k\right) \simeq -2\,[u_k(\hat{r}_s)]^2\,\bar{W}\,J_0(\epsilon)\,\sin\bar{\varphi}\,\tilde{W} - [u_k(\hat{r}_s)]^2 \times \bar{W}^2 J_0(\epsilon)\,\cos\bar{\varphi}\,\tilde{\varphi} - 2\,[u_k(\hat{r}_s)]^2\,\bar{W}^2 \times \sum_{p=1,\infty}J_p(\epsilon)\,\sin\left(\bar{\varphi}-p\,\frac{\pi}{2}\right)\cos(p\,\hat{\omega}_f\,\hat{t}), \tag{46}$$

$$\frac{d\tilde{\varphi}}{d\hat{t}} = \sum_{k=1,\infty}\tilde{h}_k. \tag{47}$$

Finally, averaging Eqs. (42)–(44) over the oscillations,[28] we obtain

$$\hat{\tau}_W\,\frac{d\bar{W}}{d\hat{t}} = \frac{\Delta'}{2\,m} + \bar{W}^{-2}J_0(\epsilon)\,\cos\bar{\varphi} + 2\,\bar{W}^{-3}J_0(\epsilon)\,\sin\bar{\varphi}\,\langle\tilde{W}\,\tilde{\varphi}\rangle - 4\,\bar{W}^{-3}\sum_{p=1,\infty}J_p(\epsilon)\,\cos\left(\bar{\varphi}-p\,\frac{\pi}{2}\right)\langle\tilde{W}\,\cos(p\,\hat{\omega}_f\,\hat{t})\rangle - 2\,\bar{W}^{-2}\sum_{p=1,\infty}J_p(\epsilon)\,\sin\left(\bar{\varphi}-p\,\frac{\pi}{2}\right)\langle\tilde{\varphi}\,\cos(p\,\hat{\omega}_f\,\hat{t})\rangle, \tag{48}$$

$$\hat{\tau}_\varphi^2\left(\frac{d\bar{h}_k}{d\hat{t}} + j_{0,k}^2\,\bar{h}_k\right) = -[u_k(\hat{r}_s)]^2\,\bar{W}^2 J_0(\epsilon)\,\sin\bar{\varphi} - 2\,[u_k(\hat{r}_s)]^2\,\bar{W}\,J_0(\epsilon)\,\cos\bar{\varphi}\,\langle\tilde{W}\,\tilde{\varphi}\rangle - 4\,[u_k(\hat{r}_s)]^2\,\bar{W} \times \sum_{p=1,\infty}J_p(\epsilon)\,\sin\left(\bar{\varphi}-p\,\frac{\pi}{2}\right)\langle\tilde{W}\,\cos(p\,\hat{\omega}_f\,\hat{t})\rangle - 2\,[u_k(\hat{r}_s)]^2\,\bar{W}^2 \times \sum_{p=1,\infty}J_p(\epsilon)\,\cos\left(\bar{\varphi}-p\,\frac{\pi}{2}\right)\langle\tilde{\varphi}\,\cos(p\,\hat{\omega}_f\,\hat{t})\rangle, \tag{49}$$

$$\frac{d\bar{\varphi}}{d\hat{t}} = \hat{\omega}_o + \sum_{k=1,\infty}\bar{h}_k. \tag{50}$$

If we assume that $\hat{\omega}_f\,\hat{\tau}_W \gg 1$ and $\hat{\omega}_f\,\hat{\tau}_\varphi^2 \gg 1$, then we can find the following approximate solutions of Eqs. (45)–(47):

$$\tilde{W} \simeq \left(\frac{2}{\hat{\omega}_f\,\hat{\tau}_W}\right)\bar{W}^{-2}\sum_{p=1,\infty}p^{-1}J_p(\epsilon)\cos\left(\bar{\varphi}-p\,\frac{\pi}{2}\right)\sin(p\,\hat{\omega}_f\,\hat{t}), \tag{51}$$

$$\tilde{\varphi} \simeq \left(\frac{2}{\hat{\omega}_f\,\hat{\tau}_\varphi^2}\right)\bar{W}^2\sum_{p=1,\infty}p^{-1}J_p(\epsilon)\left[\Sigma_c\,\cos(p\,\hat{\omega}_f\,\hat{t}) - \Sigma_s\,\sin(p\,\hat{\omega}_f\,\hat{t})\right] \times \sin\left(\bar{\varphi}-p\,\frac{\pi}{2}\right), \tag{52}$$

where

$$\Sigma_c(\hat{r}_s,\hat{\omega}_f) = \sum_{k=1,\infty}\frac{[u_k(\hat{r}_s)]^2\,p\,\hat{\omega}_f}{p^2\,\hat{\omega}_f^2 + j_{0,k}^4}, \tag{53}$$

$$\Sigma_s(\hat{r}_s,\hat{\omega}_f) = \sum_{k=1,\infty}\frac{[u_k(\hat{r}_s)]^2\,j_{0,k}^2}{p^2\,\hat{\omega}_f^2 + j_{0,k}^4}. \tag{54}$$

It follows that

$$\langle\tilde{W}\,\tilde{\varphi}\rangle \simeq -\left[\frac{2}{(\hat{\omega}_f\,\hat{\tau}_W)\,(\hat{\omega}_f\,\hat{\tau}_\varphi^2)}\right]\sum_{p=1,\infty}p^{-2}\left[J_p(\epsilon)\right]^2\Sigma_s \times \cos\left(\bar{\varphi}-p\,\frac{\pi}{2}\right)\sin\left(\bar{\varphi}-p\,\frac{\pi}{2}\right), \tag{55}$$

$$\langle\tilde{W}\,\cos(p\,\hat{\omega}_f\,\hat{t})\rangle \simeq 0, \tag{56}$$

$$\langle\tilde{\varphi}\,\cos(p\,\hat{\omega}_f\,\hat{t})\rangle \simeq \left(\frac{1}{\hat{\omega}_f\,\hat{\tau}_\varphi^2}\right)\bar{W}^2\,p^{-1}J_p(\epsilon)\,\Sigma_c\,\sin\left(\bar{\varphi}-p\,\frac{\pi}{2}\right). \tag{57}$$

Thus, to lowest order [in $(\hat{\omega}_f\,\hat{\tau}_W)^{-1}$ and $(\hat{\omega}_f\,\hat{\tau}_\varphi^2)^{-1}$], Eqs. (48)–(50) reduce to

$$\hat{\tau}_W\,\frac{d\bar{W}}{d\hat{t}} \simeq \frac{\Delta'}{2\,m} + \bar{W}^{-2}J_0(\epsilon)\,\cos\bar{\varphi} - \left(\frac{1}{\hat{\tau}_\varphi^2}\right)\left(\Sigma_o\,\cos^2\bar{\varphi} + \Sigma_e\,\sin^2\bar{\varphi}\right), \tag{58}$$

$$\hat{\tau}_\varphi^2\left(\frac{d\bar{h}_k}{d\hat{t}} + j_{0,k}^2\,\bar{h}_k\right) \simeq -[u_k(\hat{r}_s)]^2\,\bar{W}^2 J_0(\epsilon)\,\sin\bar{\varphi} + [u_k(\hat{r}_s)]^2 \times \left(\frac{1}{\hat{\tau}_\varphi^2}\right)(\Sigma_o - \Sigma_e)\bar{W}^4\,\cos\bar{\varphi}\,\sin\bar{\varphi}, \tag{59}$$

$$\frac{d\bar{\varphi}}{d\hat{t}} = \hat{\omega}_o + \sum_{k=1,\infty}\bar{h}_k, \tag{60}$$

where

$$\Sigma_o(\epsilon,\hat{r}_s,\hat{\omega}_f) = 2\sum_{p=1,\infty}^{p\,\mathrm{odd}}\left[J_p(\epsilon)\right]^2\sum_{k=1,\infty}\frac{[u_k(\hat{r}_s)]^2}{p^2\,\hat{\omega}_f^2 + j_{0,k}^4}, \tag{61}$$

$$\Sigma_e(\epsilon, \hat{r}_s, \hat{\omega}_f) = 2 \sum_{p=1,\infty}^{p\,\rm even} \left[J_p(\epsilon)\right]^2 \sum_{k=1,\infty} \frac{[u_k(\hat{r}_s)]^2}{p^2\,\hat{\omega}_f^2 + j_{0,k}^4}. \quad (62)$$

In the limit $\hat{\omega}_f \ll 1$, the previous expressions simplify to give[29]

$$\Sigma_o(\epsilon, \hat{r}_s, 0) = \left[\frac{1 - J_0(2\,\epsilon)}{2}\right] g(\hat{r}_s), \quad (63)$$

$$\Sigma_e(\epsilon, \hat{r}_s, 0) = \left[\frac{1 + J_0(2\,\epsilon)}{2} - J_0^2(\epsilon)\right] g(\hat{r}_s), \quad (64)$$

where

$$g(\hat{r}_s) = \sum_{k=1,\infty} \frac{[u_k(\hat{r}_s)]^2}{j_{0,k}^4}. \quad (65)$$

The function $g(\hat{r}_s)$ is plotted in Fig. 1. The typical variation of the functions $\Sigma_o(\epsilon, \hat{r}_s, \hat{\omega}_f)$ and $\Sigma_e(\epsilon, \hat{r}_s, \hat{\omega}_f)$ with $\hat{\omega}_f$ is illustrated in Fig. 2.

C. Phase-locked solutions

Let us search for a solution of Eqs. (58)–(60) which is such that $\bar{\varphi}$ is constant in time. This implies that, on time scales much longer than the phase-oscillation time, $1/\omega_f$, the island chain co-rotates with the external magnetic perturbation, i.e., its phase is *locked* to that of the external perturbation. We find that

$$\bar{h}_k \simeq \frac{[u_k(\hat{r}_s)]^2}{\hat{\tau}_\varphi^2\, j_{0,k}^2}\, \bar{W}^2 \left[-J_0(\epsilon)\,\sin\bar{\varphi} + \frac{(\Sigma_o - \Sigma_e)}{\hat{\tau}_\varphi^2}\,\bar{W}^2\,\cos\bar{\varphi}\,\sin\bar{\varphi}\right]. \quad (66)$$

Hence, we obtain

$$\frac{d\bar{\varphi}}{d\hat{t}} = \hat{\omega}_o + \frac{\ln(1/\hat{r}_s)}{\hat{\tau}_\varphi^2} \times \bar{W}^2 \left[-J_0(\epsilon)\,\sin\bar{\varphi} + \frac{(\Sigma_o - \Sigma_e)}{\hat{\tau}_\varphi^2}\,\bar{W}^2\,\cos\bar{\varphi}\,\sin\bar{\varphi}\right] = 0, \quad (67)$$

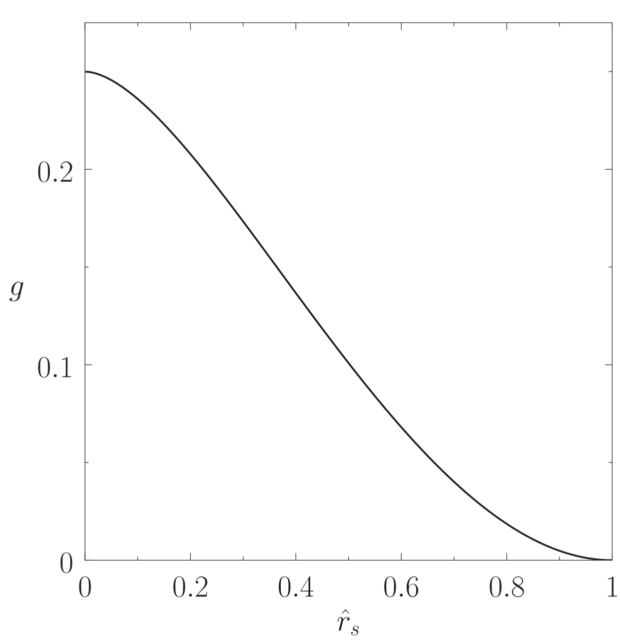

FIG. 1. The function $g(\hat{r}_s)$.

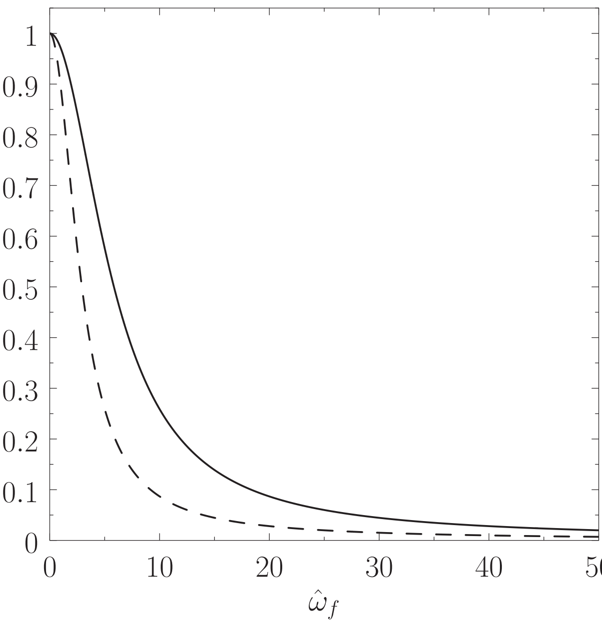

FIG. 2. The solid and dashed curves show the functions $\Sigma_o(2.4, 0.5, \hat{\omega}_f)/\Sigma_o(2.4, 0.5, 0)$ and $\Sigma_e(2.4, 0.5, \hat{\omega}_f)/\Sigma_e(2.4, 0.5, 0)$, respectively.

where use has been made of the identity[30]

$$\sum_{k=1,\infty} \frac{[u_k(\hat{r}_s)]^2}{j_{0,k}^2} = \ln\left(\frac{1}{\hat{r}_s}\right). \quad (68)$$

Equation (67) can be written as

$$f(\bar{\varphi}) \equiv \frac{\hat{\omega}_o\,\hat{\tau}_\varphi^2}{\ln(1/\hat{r}_s)} + \bar{W}^2 \left[-J_0(\epsilon)\,\sin\bar{\varphi} + \frac{(\Sigma_o - \Sigma_e)}{\hat{\tau}_\varphi^2}\,\bar{W}^2\,\cos\bar{\varphi}\,\sin\bar{\varphi}\right] = 0. \quad (69)$$

We shall refer to Eq. (69) as the *time-averaged torque balance equation* because it states that, in order for the island chain to be phase-locked to the external perturbation (on time scales much longer than $1/\omega_f$), the mean electromagnetic locking torque exerted on the plasma in the vicinity of the rational surface [i.e., the components of $f(\bar{\varphi})$ that involve $\sin\bar{\varphi}$] must balance the mean viscous restoring torque (i.e., the remaining component).[16] An examination of Eq. (67) reveals that if $\bar{\varphi}_0$ is a solution of Eq. (69), then this solution is dynamically stable provided $(df/d\bar{\varphi})_{\bar{\varphi}=\bar{\varphi}_0} < 0$ and dynamically unstable otherwise.

Finally, the *time-averaged Rutherford equation*, Eq. (58), can be written as

$$\hat{\tau}_W\,\frac{d\bar{W}}{d\hat{t}} \simeq \frac{\Delta'}{2\,m} + \bar{W}^{-2}\left[J_0(\epsilon)\,\cos\bar{\varphi} - \frac{\Sigma_o}{\hat{\tau}_\varphi^2}\,\bar{W}^2 \times \cos^2\bar{\varphi} - \frac{\Sigma_e}{\hat{\tau}_\varphi^2}\,\bar{W}^2\,\sin^2\bar{\varphi}\right], \quad (70)$$

where $\bar{\varphi}$ is a dynamically stable root of Eq. (69).

IV. RESULTS

A. Zero offset-frequency solutions

Suppose that $\hat{\omega}_o = 0$, which corresponds to the simple case in which the offset-frequency is zero, i.e., the mean angular frequency of the external magnetic perturbation matches the island chain's natural angular frequency. Let us also assume that $\hat{\omega}_f \ll 1$, for the sake of simplicity. In this case, the time-averaged torque balance equation, (69), reduces to

$$F(\bar{\varphi}) \equiv -J_0(\epsilon)\,\sin\bar{\varphi} + \left[J_0^2(\epsilon) - J_0(2\,\epsilon)\right]\delta\,\cos\bar{\varphi}\,\sin\bar{\varphi} = 0, \tag{71}$$

where

$$\delta = \frac{\bar{W}^2\,g(\hat{r}_s)}{\hat{\tau}_\varphi^2}, \tag{72}$$

and Eqs. (63) and (64) are used. A dynamically stable solution is such that $dF/d\bar{\varphi} < 0$. Furthermore, the time-averaged Rutherford equation, (70), becomes

$$\hat{\tau}_W\,\frac{d\bar{W}}{d\hat{t}} \simeq \frac{\Delta'}{2\,m} + \bar{W}^{-2}H, \tag{73}$$

where

$$H = J_0(\epsilon)\,\cos\bar{\varphi} - \left[1 - J_0^2(\epsilon)\right]\frac{\delta}{2} - \left[J_0^2(\epsilon) - J_0(2\epsilon)\right]\frac{\delta}{2}\,\cos(2\bar{\varphi}). \tag{74}$$

Note that the ordering assumptions that we have made during the derivation of Eqs. (71) and (74) constrain the quantity $\bar{W}^2/\hat{\tau}_\varphi^2$ and, hence, the parameter δ, to be much smaller than unity.

Defining $s = \mathrm{sgn}[J_0(\epsilon)]$ and

$$x = \frac{|J_0(\epsilon)|}{\delta\,|J_0^2(\epsilon) - J_0(2\,\epsilon)|}, \tag{75}$$

the appropriate solution of Eq. (71) is

$$\begin{aligned}
&\cos\bar{\varphi} = +1, && \sin\bar{\varphi} = 0 && x > 1,\ s > 0,\\
&\cos\bar{\varphi} = -1, && \sin\bar{\varphi} = 0 && x > 1,\ s < 0,\\
&\cos\bar{\varphi} = +x, && \sin\bar{\varphi} = +\sqrt{1-x^2} && x < 1,\ s > 0,\\
&\cos\bar{\varphi} = -x, && \sin\bar{\varphi} = -\sqrt{1-x^2} && x < 1,\ s < 0.
\end{aligned} \tag{76}$$

Figures 3 and 4 illustrate the behavior of this solution when $\delta \ll 1$. It can be seen that the island chain's helical phase matches the mean phase of the external perturbation (i.e., $\bar{\varphi} = 0$) when the phase-oscillation amplitude, ϵ, is such that $J_0(\epsilon) > 0$. On the other hand, the two phases differ by π radians (i.e., $\bar{\varphi} = -\pi$) when the phase-oscillation amplitude is such that $J_0(\epsilon) < 0$. Unfortunately, in both cases, the phase relation is destabilizing (i.e., $H > 0$). However, if the phase-oscillation amplitude is such that $|J_0(\epsilon)| \ll 1$, then the terms involving the small parameter δ that appear in Eqs. (71) and (74) become important. It can be seen that these terms cause the island chain to lock to the mean phase of the external perturbation in phase quadrature (i.e., $\bar{\varphi} = \pm\pi/2$.) Moreover, this phase relation is stabilizing (i.e., $H < 0$). It is evident that the range of ϵ values around the maximally stabilizing value $\epsilon = j_{0,k}$ (where $j_{0,k}$ is the kth zero of the J_0 Bessel function) for which the phase relation between the island chain and the external perturbation is stabilizing becomes larger as the parameter δ increases and approaches zero as $\delta \to 0$.

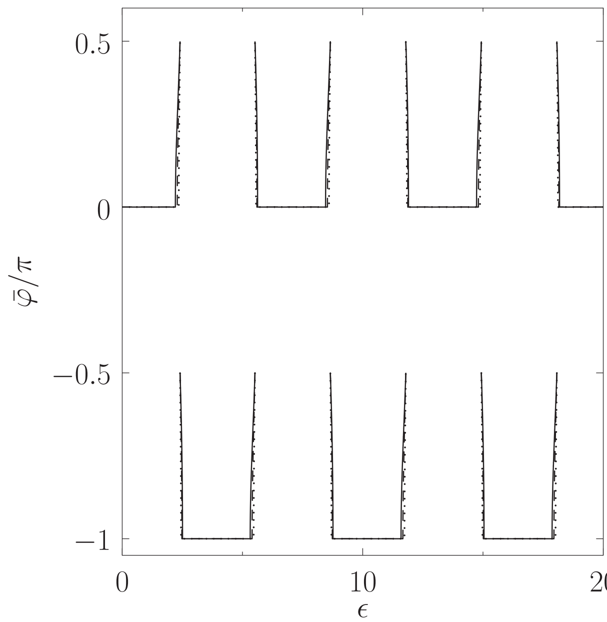

FIG. 3. Zero offset-frequency solutions. The dotted, dashed, and solid curves show the mean locking phase, $\bar{\varphi}$, plotted as a function of the phase-oscillation amplitude, ϵ, for $\delta = 0.1$, 0.2, and 0.4, respectively.

B. Finite offset-frequency solutions

Suppose that the offset-frequency, ω_o, is non-zero. Let us again assume that $\hat{\omega}_f \ll 1$, for the sake of simplicity. In this case, the time-averaged torque balance equation reduces to

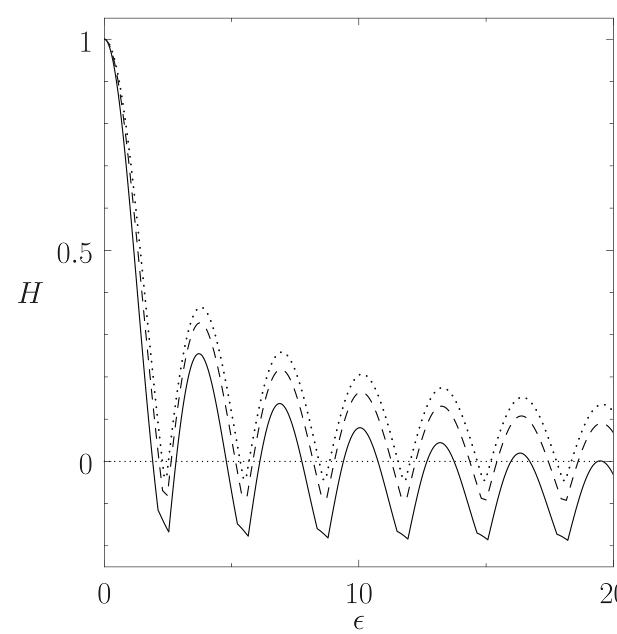

FIG. 4. Zero offset-frequency solutions. The dotted, dashed, and solid curves show the island stability parameter, H, plotted as a function of the phase-oscillation amplitude, ϵ, for $\delta = 0.1$, 0.2, and 0.4, respectively.

$$F(\bar{\varphi}) \equiv \zeta - J_0(\epsilon)\,\sin\bar{\varphi} + \left[J_0^2(\epsilon) - J_0(2\,\epsilon)\right]\delta\,\cos\bar{\varphi}\,\sin\bar{\varphi} = 0, \tag{77}$$

where

$$\zeta = \frac{\hat{\omega}_o\,\hat{\tau}_\varphi^2}{\ln(1/\hat{r}_s)\,\bar{W}^2}. \tag{78}$$

The time-averaged Rutherford equation again takes the form (73). As before, a dynamically stable solution of Eq. (77) is such that $dF/d\bar{\varphi} < 0$.

Equation (77) can be solved numerically as follows: the cosine of $\bar{\varphi}$ is a real root of the polynomial

$$\left[J_0^2(\epsilon)-J_0(2\epsilon)\right]^2\delta^2\cos^4\bar{\varphi} - 2J_0(\epsilon)\left[J_0^2(\epsilon)-J_0(2\epsilon)\right]\delta\cos^3\bar{\varphi} + \left\{J_0^2(\epsilon) - \left[J_0^2(\epsilon)-J_0(2\epsilon)\right]^2\delta^2\right\}\cos^2\bar{\varphi} + 2J_0(\epsilon) \times \left[J_0^2(\epsilon)-J_0(2\epsilon)\right]\delta\cos\bar{\varphi} + \zeta^2 - J_0^2(\epsilon) = 0; \tag{79}$$

the sine of $\bar{\varphi}$ is then given by

$$\sin\bar{\varphi} = \frac{\zeta}{J_0(\epsilon) - \left[J_0^2(\epsilon) - J_0(2\,\epsilon)\right]\delta\,\cos\bar{\varphi}}. \tag{80}$$

A valid solution is such that $|\cos\bar{\varphi}| < 1$ and $dF/d\bar{\varphi} < 0$.

Figures 5 and 6 illustrate the behavior of the solution to the previous two equations when $\delta = 0.4$. It can be seen that if the frequency-offset parameter, ζ, is very much less than unity, then the finite offset-frequency solutions are similar in form to the zero offset-frequency solutions described in Sec. IV A. However, as ζ increases, ranges of ϵ values, centered on the values $\epsilon = j_{0,k}$, develop in which there are no phase-locked solutions. Physically, this is because the mean electromagnetic locking torque is too small to balance the viscous restoring torque in the time-averaged torque balance equation. It is clear that in the limit $\zeta \rightarrow 1$, phase-locked solutions only exist when then phase-oscillation amplitude, ϵ, is relatively small, and these solutions are such that the island chain is destabilized by the external perturbation. Thus, we conclude that the result, obtained in Sec. IV A, that it is possible to phase-lock an island chain in a stabilizing phase relation with an external perturbation of rapidly oscillating phase only holds when the offset-frequency is relatively small (i.e., $\zeta \lesssim 0.4$).

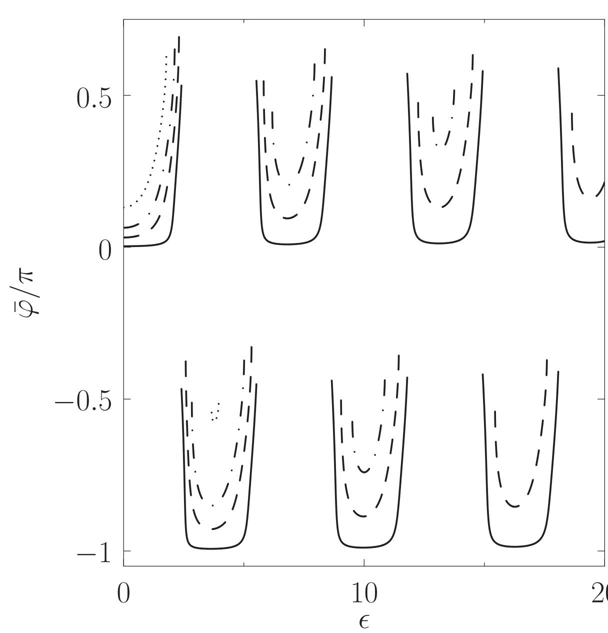

FIG. 5. Finite offset-frequency solutions. The mean locking phase, $\bar{\varphi}$, plotted as a function of the phase-oscillation amplitude, ϵ, for $\delta = 0.4$. The solid, dashed, dashed-dotted, and dotted curves correspond to $\zeta = 0.01$, 0.1, 0.2, and 0.4, respectively.

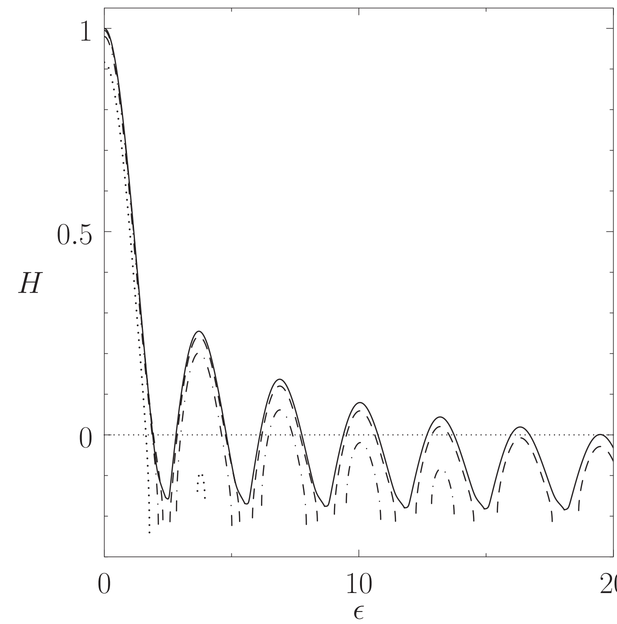

FIG. 6. Finite offset-frequency solutions. The island stability parameter, H, plotted as a function of the phase-oscillation amplitude, ϵ, for $\delta = 0.4$. The solid, dashed, dashed-dotted, and dotted curves correspond to $\zeta = 0.01$, 0.1, 0.2, and 0.4, respectively.

V. SUMMARY AND CONCLUSIONS

The analogy that exists between the phase evolution of a magnetic island chain interacting with an external magnetic perturbation of the same helicity and the angular motion of a rigid pendulum led us to speculate (in Sec. I) that if the helical phase of the external perturbation were subject to a high frequency oscillation, then it might be possible to induce the island chain to lock to the perturbation in a stabilizing phase relation. In order to explore this possibility, the dynamics of a magnetic island chain interacting with a magnetic perturbation of rapidly oscillating phase is analyzed using Kapitza's method in Sec. III. The phase oscillations are found to modify the existing terms and also to give rise to new terms, in the equations governing the secular evolution of the island chain's radial width and helical phase—see Eqs. (58)–(60). An examination of the properties of the new secular evolution equations (see Sec. IV) reveals that it is indeed possible to phase-lock an island chain to an external magnetic perturbation with an oscillating helical phase in a *stabilizing* phase relation provided that the amplitude, ϵ, of the phase oscillations (in radians) is such that $|J_0(\epsilon)| \ll 1$, and the mean angular frequency of the perturbation closely matches the natural angular frequency of the island chain.

ACKNOWLEDGMENTS

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[24]*Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, edited by M. Abramowitz and I. A. Stegun (Dover, 1965), Eqs. 11.4.5 and 9.1.28.
[25]If $f(\hat{r})$ is a general well-behaved function in the domain $0 \leq \hat{r} \leq 1$, then we can write $f(\hat{r}) = \sum_{k=1,\infty} f_k\, u_k(\hat{r})$, where, from Eq. (26), $f_k = \int_0^1 \hat{r}\, f(\hat{r})\, u_k(\hat{r})\, d\hat{r}$. Let $g(\hat{r}, \hat{r}_s) = \sum_{k=1,\infty} \hat{r}\, u_k(\hat{r}_s)\, u_k(\hat{r})$. It follows from Eq. (26) that $\int_0^1 f(\hat{r})\, g(\hat{r}, \hat{r}_s)\, d\hat{r} = \sum_{k=1,\infty} f_k\, u_k(\hat{r}_s) = f(\hat{r}_s)$. Because the previous result is true for general (well-behaved) $f(\hat{r})$, and for general $\hat{r}_s$ (provided that $0 \leq \hat{r}_s \leq 1$), we deduce that $g(\hat{r}, \hat{r}_s) = \delta(\hat{r} - \hat{r}_s)$.
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[30]As is easily demonstrated, the time-independent solution of Eqs. (22) and (23) is such that $\Omega(\hat{r}_s) = -\ln(1/\hat{r}_s)\, \hat{W}^2\, \sin(\varphi_s - \varphi_v)/\hat{\tau}_\varphi^2$. However, the time-independent solution of the equivalent set of equations, (28) and (29), yields $\Omega(\hat{r}_s) = -\sum_{k=1,\infty}([u_k(\hat{r}_s)]^2/j_{0,k}^2)\, \hat{W}^2\, \sin(\varphi_s - \varphi_v)/\hat{\tau}_\varphi^2$. Hence, we deduce that $\ln(1/\hat{r}_s) = \sum_{k=1,\infty}([u_k(\hat{r}_s)]^2/j_{0,k}^2)$.