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A mathematical formalism is developed for calculating the ion polarization term in the Rutherford island width evolution equation in the presence of drift-acoustic waves. The calculation is fully nonlinear, includes both ion and electron diamagnetic effects, as well as ion compressibility, but is performed in slab geometry. Magnetic islands propagating in a certain range of phase velocities are found to emit drift-acoustic waves. Wave emission gives rise to rapid oscillations in the ion polarization term as the island phase velocity varies, and also generates a net electromagnetic force acting on the island region. Increasing ion compressibility is found to extend the range of phase velocities over which drift-acoustic wave emission occurs in the electron diamagnetic direction.
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I. INTRODUCTION

Tearing modes are magnetohydrodynamic (MHD) instabilities that often limit fusion plasma performance in magnetic confinement devices, such as tokamaks, which rely on the existence of nested toroidal magnetic flux surfaces.[1] As the name suggests, "tearing" modes tear and reconnect magnetic field lines, in the process converting nested toroidal flux surfaces into helical magnetic islands. Such islands degrade plasma confinement because heat and particles are able to travel radially from one side of an island to another by flowing along magnetic field lines, which is a relatively fast process, instead of having to diffuse across magnetic flux surfaces, which is a relatively slow process.[2]

In two-fluid MHD theory, which is far more relevant to present-day magnetic confinement devices than single-fluid MHD theory, a magnetic island is embedded within ion and electron fluids that generally flow at different velocities. The island itself usually propagates at some intermediate velocity. For a sufficiently wide island (i.e., much wider than the collisionless ion skin depth), both fluids are required to flow at the island propagation velocity in the region lying within the magnetic separatrix (since neither fluid can easily cross the separatrix, or any other magnetic flux surface). However, the region immediately outside the separatrix is characterized by strongly sheared ion and electron fluid flow profiles, as the velocities of both fluids adjust to their unperturbed values far away from the island. The polarization current generated by the strongly sheared ion flow around the island separatrix gives rise to an additional term[3] in the well-known Rutherford island width evolution equation,[4] which takes the normalized (see Sec. II A) form

$$\frac{0.823}{\eta}\frac{dW}{dt} = \Delta' + \Delta_{\rm polz} \tag{1}$$

in slab geometry. Here, W is the full island width, η is the local plasma resistivity, and Δ' is the conventional tearing stability index.[5] The standard expression for the ion polarization term (for wide islands) is[6,7]

$$\Delta_{\rm polz} = 1.38\frac{(V-V_{EB})(V-V_i)}{(W/4)^3}, \tag{2}$$

where V is the island phase velocity, V_{EB} is the unperturbed (i.e., in the absence of an island) local $\mathbf{E}\times\mathbf{B}$ velocity, and V_i is the unperturbed local ion fluid velocity. The polarization term can have a significant influence on magnetic island stability in present-day magnetic confinement devices, and is stabilizing when the island phase velocity lies between the unperturbed local $\mathbf{E}\times\mathbf{B}$ and ion fluid velocities, and destabilizing otherwise.[8]

According to Waelbroeck *et al.*,[8] a magnetic island propagating in a certain range of phase velocities emits drift waves. This effect is not taken into account in the standard derivation of the ion polarization term. Moreover, Smolyakov *et al.*[9] have recently claimed that ion compressibility, which is also not taken into account in the standard derivation of the ion polarization term, gives rise to an additional stabilizing term in the Rutherford island width evolution equation. This term varies as c_s^2/W, where c_s is the local sound speed.

The aim of this paper is to investigate the effect of drift-acoustic waves (i.e., drift waves modified by ion compressibility) on the stability of magnetic islands. In other words, we wish to determine whether drift-acoustic waves give rise to any modifications to the standard expression, (2), for the ion polarization term in the Rutherford island width evolution equation, or to any additional terms in this equation. For the sake of simplicity, our analysis is restricted to slab geometry. However, unlike Ref. 8, we employ a fully nonlinear and compressible description of the ion dynamics.

II. ANALYSIS

A. Reduced equations

Standard right-handed Cartesian coordinates (x,y,z) are adopted. Consider a quasineutral plasma with singly charged ions of mass m_i. The ion/electron number density n_0 is assumed to be uniform and constant. Suppose that $T_i=\tau T_e$, where $T_{i,e}$ is the ion/electron temperature, and τ is uniform and constant. Let there be no variation of quantities in the z

direction: i.e., $\partial/\partial z \equiv 0$. Finally, let all lengths be normalized to a, which is the typical variation length scale of B_y; let all magnetic field strengths be normalized to B_a, which is the typical strength of B_y away from the resonant surface; and let all times be normalized to a/V_a, where $V_a = B_a/\sqrt{\mu_0 n_0 m_i}$.

We can write $\mathbf{B} = \nabla\psi \times \hat{\mathbf{z}} + (B_0 + b_z)\hat{\mathbf{z}}$ and $P = P_0 - B_0 b_z + O(1)$, where $\mathbf{B}$ is the magnetic field and P is the total plasma pressure. Here, we are assuming that P_0 and B_0 are uniform, and $P_0 \gg B_0 \gg 1$, with ψ and b_z both $O(1)$.[7] When expressed in unnormalized quantities, our fundamental ordering takes the form

$$\frac{\mu_0 P_0}{B_a^2} \gg \frac{B_0}{B_a} \gg 1. \tag{3}$$

Let $\beta = \Gamma P_0/B_0^2$ be (Γ times) the plasma beta calculated with the "guide field," B_0, where $\Gamma = 5/3$ is the plasma ratio of specific heats. Note that the above ordering scheme (which is effectively a high poloidal beta scheme, i.e., $\mu_0 P_0/B_a^2 \gg 1$, in unnormalized units) does not constrain β (which is effectively the toroidal beta) to be either much less than or much greater than unity.

We adopt the reduced, two-dimensional (2D), drift-MHD equations derived in Ref. 7 (neglecting some unimportant terms involving μ_e for the sake of simplicity),

$$\frac{\partial\psi}{\partial t} = [\phi, \psi] - [Z, \psi] + \eta J, \tag{4}$$

$$\frac{\partial Z}{\partial t} = [\phi, Z] + c_\beta^2 [V_z, \psi] + d_\beta^2 [J, \psi] + D\nabla^2 Z, \tag{5}$$

$$\frac{\partial \nabla^2\phi}{\partial t} = [\phi, \nabla^2\phi] - \frac{\tau}{2}(\nabla^2[\phi, Z] + [\nabla^2\phi, Z] + [\nabla^2 Z, \phi]) + [J, \psi] + \mu_i \nabla^4(\phi + \tau Z) + \mu_e \nabla^4(\phi - Z), \tag{6}$$

$$\frac{\partial V_z}{\partial t} = [\phi, V_z] + [Z, \psi] + \mu_i \nabla^2 V_z, \tag{7}$$

where $J = 1 + \nabla^2\psi$ and $D = c_\beta^2\eta + (1 - c_\beta^2)\kappa$. Here, $c_\beta = \sqrt{\beta/(1+\beta)}$, $d_\beta = c_\beta d_i/\sqrt{1+\tau}$, $Z = d_i b_z/(1+\tau)$, $d_i = (m_i/n_0 e^2 \mu_0)^{1/2}/a$, and $[A, B] = \nabla A \times \nabla B \cdot \hat{\mathbf{z}}$. Of course, d_i is the (normalized) collisionless ion skin depth. The (normalized) length d_β reduces to the (normalized) collisionless ion skin depth, d_i, in the limit $\beta \gg 1$, and to the (normalized) sonic ion gyroradius, $\rho_s = \sqrt{\beta} d_i$, in the limit $\beta \ll 1$. Moreover, the parameter c_β reduces to unity in the limit $\beta \gg 1$, and to the ion sound speed, $c_s = \sqrt{\beta}$, in the limit $\beta \ll 1$. The guiding-center velocity is written $\mathbf{V}_{EB} \simeq \nabla\phi \times \hat{\mathbf{z}} + d_i (c_\beta/d_\beta)^2 V_z \hat{\mathbf{z}}$. The ion fluid velocity is written $\mathbf{V}_i \simeq \nabla(\phi + \tau Z) \times \hat{\mathbf{z}} + d_i (c_\beta/d_\beta)^2 V_z \hat{\mathbf{z}}$. Finally, the electron fluid velocity is written $\mathbf{V}_e \simeq \nabla(\phi - Z) \times \hat{\mathbf{z}} + d_i (c_\beta/d_\beta)^2 [V_z + (d_\beta/c_\beta)^2 J]\hat{\mathbf{z}}$. Note that the neglect of V_z is equivalent to the assumption that the ion fluid is incompressible. Furthermore, η is the (uniform) plasma resistivity, $\mu_{i,e}$ the (uniform) ion/electron viscosity, and κ the (uniform) plasma thermal conductivity. The above equations contain both electron and ion diamagnetic effects, including the contribution of the anisotropic ion gyroviscous tensor, but neglect electron inertia. Our equations are "reduced" in the sense that they do not contain the compressible Alfvén wave.

B. Island geometry

Consider a slab plasma that is periodic in the y direction with periodicity length l. Let the system be symmetric about $x = 0$. i.e., $\psi(-x, y, t) = \psi(x, y, t)$, $Z(-x, y, t) = -Z(x, y, t)$, $\phi(-x, y, t) = -\phi(x, y, t)$, $V_z(-x, y, t) = V_z(x, y, t)$, and $J(-x, y, t) = J(x, y, t)$. Consider an isolated, saturated, constant-ψ magnetic island, centered on $x = 0$. It is convenient to transform to the island rest frame, in which $\partial/\partial t \equiv 0$. Suppose that the island is embedded in a plasma with uniform (but different) y-directed ion and electron fluid velocities.

In the immediate vicinity of the island, we can write

$$\psi(x, \theta) = -\frac{x^2}{2} + \Psi \cos\theta, \tag{8}$$

where $\theta = ky$, $k = 2\pi/l$, and $\Psi > 0$ is the reconnected magnetic flux. As is well known, the above expression for ψ describes a "cat's eye" magnetic island of full width (in the x direction) $W = 4w$, where $w = \sqrt{\Psi}$. The region inside the magnetic separatrix corresponds to $\Psi \geq \psi > -\Psi$, whereas the region outside the separatrix corresponds to $\psi < -\Psi$.

It is helpful to define a flux-surface average operator,

$$\langle f(s, \psi, \theta) \rangle = \oint \frac{f(s, \psi, \theta)}{|x|} \frac{d\theta}{2\pi} \tag{9}$$

for $\psi < -\Psi$, and

$$\langle f(s, \psi, \theta) \rangle = \int_{-\theta_0}^{\theta_0} \frac{f(s, \psi, \theta) + f(-s, \psi, \theta)}{2|x|} \frac{d\theta}{2\pi} \tag{10}$$

for $\Psi \geq \psi > -\Psi$. Here, $s = \mathrm{sgn}(x)$ and $x(s, \psi, \theta_0) = 0$ (with $\pi > \theta_0 > 0$). The most important property of this operator is that $\langle [A, \psi] \rangle \equiv 0$ for any field $A(s, \psi, \theta)$. Let $\widetilde{A} \equiv A - \langle A \rangle / \langle 1 \rangle$.

C. Ion polarization term

As is well known, the ion polarization term in the Rutherford island width evolution equation is generated from the component of the parallel electric current, $J(\psi, \theta)$, with the symmetry of $\cos\theta$:[7]

$$\Delta_{\mathrm{polz}} = \frac{4}{\Psi} \int_{\Psi}^{-\infty} \langle J \cos\theta \rangle d\psi. \tag{11}$$

Furthermore, any component of the current with the symmetry of $\sin\theta$ gives rise to a net electromagnetic force acting on the island region in the y direction,[10]

$$F_y = -2k\Psi \int_{\Psi}^{-\infty} \langle J \sin\theta \rangle d\psi. \tag{12}$$

D. Ordering scheme

Let $\eta = \epsilon\hat{\eta}$, $D = \epsilon\hat{D}$, and $\mu_{ie} = \epsilon\hat{\mu}_{i,e}$, where the hatted terms are all $O(1)$, and where

$$\epsilon \ll d_\beta^2 \ll 1. \tag{13}$$

The above ordering scheme assumes that the (unnormalized) length d_β is much smaller than the typical variation length scale of the equilibrium, and that transport terms are extremely small compared to the other terms in our equations. These assumptions are appropriate to high-temperature fusion plasmas.

E. Analysis

In the island rest frame, the flux-surface average of Eq. (4) yields

$$\langle J\rangle = 0. \tag{14}$$

In the following, we shall neglect transport [i.e., $O(\epsilon)$] terms all together. Hence, in the island rest frame, Eqs. (4)–(7) reduce to

$$0 = [\phi,\psi] - [Z,\psi], \tag{15}$$

$$0 = [\phi,Z] + c_\beta^2[V_z,\psi] + d_\beta^2[J,\psi], \tag{16}$$

$$0 = [\phi,\nabla^2\phi] - \frac{\tau}{2}(\nabla^2[\phi,Z] + [\nabla^2\phi,Z] + [\nabla^2 Z,\phi]) + [J,\psi], \tag{17}$$

$$0 = [\phi,V_z] + [Z,\psi]. \tag{18}$$

All quantities in the above equations are assumed to be $O(1)$, except for the small parameter d_β^2.

Let

$$\phi(x,\theta) = s[\phi_0(\psi) + d_\beta^2\phi_1(\psi,\theta)], \tag{19}$$

$$Z(x,\theta) = s[Z_0(\psi) + d_\beta^2 Z_1(\psi,\theta)], \tag{20}$$

$$V_z(x,\theta) = V_0(\psi) + d_\beta^2 V_1(\psi,\theta). \tag{21}$$

To lowest order in d_β^2, Eqs. (15)–(18) yield

$$0 \simeq [\phi_1,\psi] - [Z_1,\psi], \tag{22}$$

$$0 \simeq [\phi_0,Z_1] + [\phi_1,Z_0] + c_\beta^2[V_1,\psi] + [J,\psi], \tag{23}$$

$$0 \simeq [\phi_0 + (\tau/2)Z_0,\nabla^2\phi_0] + (\tau/2)[\phi_0,\nabla^2 Z_0] + [\phi_0 + \tau Z_0, d_\beta^2\nabla^2\phi_1] + [J,\psi], \tag{24}$$

$$0 \simeq [\phi_1,V_0] + [\phi_0,V_1] + [Z_1,\psi]. \tag{25}$$

Ion compressibility and drift waves enter our analysis via the terms involving c_β^2 and $d_\beta^2\nabla^2\phi_1$, respectively, in the above set of equations. The retention of the $d_\beta^2\nabla^2\phi_1$ term is consistent with our ordering scheme, since drift waves cause ϕ_1 to vary on an $O(d_\beta)$ length scale in the x direction.

Let $M(\psi)=d\phi_0/d\psi$, $L(\psi)=dZ_0/d\psi$, and $N(\psi)=dV_0/d\psi+1$. Note that $M(\psi)$ and $L(\psi)$ must both be zero inside the island separatrix, since it is impossible to have a non zero odd (in x) flux-surface function in this region. The (normalized) $\mathbf{E}\times\mathbf{B}$, ion, and electron fluid velocities in the y direction are given by $V_{EB}=|x|M$, $V_i=|x|(M+\tau L)$, and $V_e=|x|(M-L)$, respectively.

After some analysis, Eqs. (14) and (22)–(25) yield $\tilde{\phi}_1=\tilde{Z}_1=M\tilde{V}_1/N$ and

$$J \simeq [M - L - c_\beta^2 N/M]\tilde{\phi}_1, \tag{26}$$

where

$$(M+\tau L)d_\beta^2\nabla^2\phi_1 - [M - L - c_\beta^2 N/M]\phi_1 \simeq -\frac{\widetilde{x^2}}{2}\frac{d[M(M+\tau L)]}{d\psi}. \tag{27}$$

Appendix A presents an alternative derivation of Eqs. (26) and (27).

F. Renormalized equations

In the immediate vicinity of the island, it is helpful to perform the following renormalization: $\hat{\psi}=-\psi/\Psi$, $X=x/w$, $\langle\langle\cdots\rangle\rangle=\langle\cdots\rangle w$, $\hat{L}=L/L_0$, $\hat{M}=M/L_0$, and $Y=\phi_1/L_0$, where $L_0=V_*/[w(1+\tau)]$. Here, V_* is the local diamagnetic velocity in the absence of an island. This renormalization is consistent with our previous ordering scheme provided $\Psi\equiv w^2$, w, and V_*/w can all be regarded as $O(1)$ compared to d_β^2. This is easily satisfied, but implies that

$$d_\beta \ll w, \tag{28}$$

i.e., the island width is much greater than d_β.

In renormalized variables,

$$X^2 = 2(\hat{\psi} + \cos\theta) \tag{29}$$

and

$$\langle\langle\cdots\rangle\rangle = \frac{1}{\pi}\int_0^\pi \frac{(\cdots)}{X}d\theta \tag{30}$$

for even functions of θ. Of course, odd functions of θ flux surface average to zero.

The ion polarization term in the Rutherford island width evolution equation, and the net electrostatic force acting on the island region, take the forms

$$\Delta_{\rm polz} = \mathcal{C}\frac{V_*^2}{w^3(1+\tau)^2} \tag{31}$$

and

$$F_y = \mathcal{D}\frac{kwV_*^2}{(1+\tau)^2}, \tag{32}$$

respectively, where the numerical factors $\mathcal{C}$ and $\mathcal{D}$ are given by

$$\mathcal{C} = -4\int_{-1}^{\infty}[\hat{M} - \hat{L} - \alpha^2 N/\hat{M}]\langle\langle\tilde{Y}\cos\theta\rangle\rangle d\hat{\psi} \tag{33}$$

and

$$\mathcal{D} = 2\int_{-1}^{\infty} [\hat{M} - \hat{L} - \alpha^2 N/\hat{M}]\langle\langle\tilde{Y}\sin\theta\rangle\rangle d\hat{\psi}, \tag{34}$$

respectively. Here,

$$\alpha = \frac{c_\beta w(1+\tau)}{V_*} \tag{35}$$

parametrizes the magnitude of ion compressibility effects.

The function $Y(X,\theta)$ satisfies

$$(\hat{M} + \tau\hat{L})d^2\frac{\partial^2 Y}{\partial X^2} - [\hat{M} - \hat{L} - \alpha^2 N/\hat{M}]Y \simeq \frac{\widetilde{X^2}}{2}\frac{d[\hat{M}(\hat{M}+\tau\hat{L})]}{d\hat{\psi}}, \tag{36}$$

where

$$d = d_\beta/w \tag{37}$$

parametrizes the magnitude of drift-wave effects. In deriving Eq. (36), we have assumed that $\partial/\partial x \gg k$ and $|\partial \ln Y/\partial x| \sim O(d_\beta^{-1})$.

G. Profile functions

The functions $\hat{M}(\hat{\psi})$ and $\hat{L}(\hat{\psi})$ satisfy the following matching conditions as $X \to \infty$:[7]

$$X\hat{M} \to -v, \tag{38}$$

$$X\hat{L} \to 1. \tag{39}$$

Here,

$$v = \frac{(V - V_{EB})(1+\tau)}{V_*}, \tag{40}$$

where V is the island phase-velocity, V_{EB} is the local $\mathbf{E}\times\mathbf{B}$ velocity in the absence of an island, and V_* is the local diamagnetic velocity in the absence of an island. Thus, $v=\tau$ corresponds to the island propagating with the local ion fluid, $v=0$ to the island propagating at the local $\mathbf{E}\times\mathbf{B}$ velocity, and $v=-1$ to the island propagating with the local electron fluid.

In the following, we adopt the following analytic forms for $\hat{M}(\hat{\psi})$, $\hat{L}(\hat{\psi})$, and $\hat{N}(\hat{\psi})$:

$$\hat{M}(\hat{\psi}) = -v\hat{L}(\hat{\psi}), \tag{41}$$

and

$$\hat{L}(\hat{\psi} < 1) = \frac{H(\hat{\psi})}{\langle\langle X^2\rangle\rangle_{\hat{\psi}=1}}, \tag{42}$$

$$\hat{L}(\hat{\psi} \geqslant 1) = \frac{H(\hat{\psi})}{\langle\langle X^2\rangle\rangle}, \tag{43}$$

and

$$N(\hat{\psi}) = [H(\hat{\psi})]^2, \tag{44}$$

where

$$H(\hat{\psi}) = \frac{1}{2}\{1 + \tanh[(\hat{\psi}-1)/d]\}. \tag{45}$$

The above forms are partly suggested by the analytic and numerical results of Refs. 7–11, and partly by the need to prevent the ion compressibility term in Eq. (36) from blowing up inside the separatrix.

Using the above profile functions, Eq. (36) reduces to

$$d^2\frac{\partial^2 Y}{\partial X^2} - \left[\frac{v + 1 - (\alpha^2/v)N(\hat{\psi})/\hat{L}^2(\hat{\psi})}{v - \tau}\right]Y \simeq -v\widetilde{X^2}\frac{d\hat{L}(\hat{\psi})}{d\hat{\psi}}, \tag{46}$$

while Eqs. (33) and (34) yield

$$\mathcal{C} = 4\int_1^{\infty} [v + 1 - (\alpha^2/v)N(\hat{\psi})/\hat{L}^2(\hat{\psi})]\hat{L}(\hat{\psi}) \times\langle\langle\tilde{Y}(X,\theta)\cos\theta\rangle\rangle d\hat{\psi} \tag{47}$$

and

$$\mathcal{D} = -2\int_1^{\infty} [v + 1 - (\alpha^2/v)N(\hat{\psi})/\hat{L}(\hat{\psi})^2]\hat{L}(\hat{\psi}) \times\langle\langle\tilde{Y}(X,\theta)\sin\theta\rangle\rangle d\hat{\psi}, \tag{48}$$

respectively.

H. Standard result

If we neglect drift waves in Eq. (46), by setting the parameter d to zero, then this equation can easily be solved to give

$$\tilde{Y}(X,\theta) = v(v-\tau)\frac{\widetilde{X^2}d\hat{L}(\hat{\psi})/d\hat{\psi}}{[v + 1 - (\alpha^2/v)/\hat{L}^2(\hat{\psi})]}. \tag{49}$$

It immediately follows from Eq. (48) and symmetry that $\mathcal{D}=0$. In other words, in the absence of drift waves, there is zero net electromagnetic force acting on the island region. Equations (29), (42), (43), (47), and (49) yield

$$\mathcal{C} = v(v-\tau)[\langle\langle\widetilde{X^2}\,\widetilde{X^2}\rangle\rangle\langle\langle X^2\rangle\rangle^{-2}]_{\hat{\psi}=1} + v(v-\tau)\int_1^{\infty}\langle\langle\widetilde{X^2}\,\widetilde{X^2}\rangle\rangle \times\frac{d\langle\langle X^2\rangle\rangle^{-2}}{d\hat{\psi}}d\hat{\psi}, \tag{50}$$

giving

$$\mathcal{C} = 1.38v(v-\tau). \tag{51}$$

This is the standard result [see Eqs. (2), (31), and (40)].[6] Note that the above expression is completely independent of the ion compressibility parameter α. Thus, we conclude that ion compressibility alone has no effect on the conventional expression, (2), for the ion polarization term in the Rutherford island width evolution equation.

I. Boundary conditions

The first boundary condition on $Y(X,\theta)$ is simply

$$Y(0,\theta) = 0, \tag{52}$$

since $Y(X,\theta)$ is an odd function of X. The second boundary condition is obtained by considering the behavior of $Y(X,\theta)$ as $X\to\infty$. There are two possible types of behavior. If the term in square brackets in Eq. (46) is positive, then the solutions grow and decay exponentially as $X\to\infty$. This corresponds to an island that is not emitting waves. Of course, the physical solution is obtained by excluding the exponentially growing solution. On the other hand, if the term in square brackets in Eq. (46) is negative, then the solutions oscillate as $X\to\infty$. This corresponds to an island that is emitting waves. In this case, the correct boundary condition is that the solution should correspond to an outgoing wave as $X\to\infty$.[12]

We adopt the following ansatz for applying the outgoing wave boundary condition to the solution of Eq. (46). First of all, we modify Eq. (46) by introducing a small complex damping term that causes the outgoing wave solution to decay and the incoming wave solution to blow up, as $X\to\infty$. Thus, the exclusion of the incoming wave solution becomes equivalent to the requirement that $Y(X,\theta)$ be well behaved as $X\to\infty$. The modification to Eq. (46) is suggested by linear analysis (see Appendix B). Thus, the modified form of Eq. (46) is

$$d^2\frac{\partial^2 Y}{\partial X^2} - \left[\frac{v+1-(\alpha^2/v)N(\hat{\psi})/\hat{L}^2(\hat{\psi})+i\nu}{v-\tau}\right]Y \simeq -v\widetilde{X^2}\,\frac{d\hat{L}(\hat{\psi})}{d\hat{\psi}}, \tag{53}$$

where $0<\nu\ll 1$.

The solution of the above equation, subject to the boundary condition (52), and to the boundary condition that $Y(X,\theta)$ be well-behaved as $X\to\infty$, yields the correct form for $Y(X,\theta)$ in the absence of wave emission, but yields a *complex* $Y(X,\theta)$ in the presence of wave emission. As explained in the following section, the complex part of $Y(X,\theta)$ can be interpreted as a component of the physical $Y(X,\theta)$ with the symmetry of $\sin\theta$.

J. Interpretation of complex solutions

In general, the solution of Eq. (53) subject to the appropriate boundary conditions yields a complex $Y(X,\theta)$ of the form

$$Y^{(c)}(X,\theta) = \sum_{m=0}^{\infty} c_m(X)e^{ik_X(X)X}\cos(m\theta) = \sum_{m=0}^{\infty} d_m(X)\cos(m\theta), \tag{54}$$

where the c_m are complex, and $d_m=c_m\exp(ik_X X)$. However, we require a real $Y(X,\theta)$, characterized by outgoing waves as $X\to\infty$, of the form

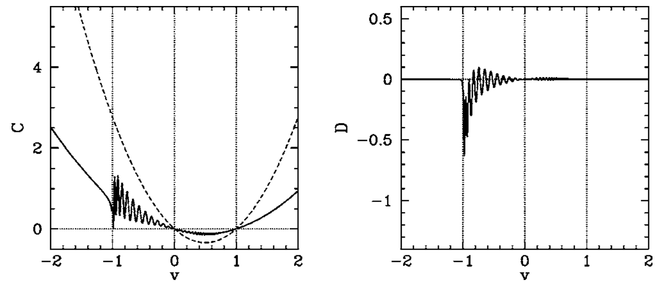

FIG. 1. The stability parameter, $\mathcal{C}$, and the force parameter, $\mathcal{D}$, calculated as functions of the island phase-velocity parameter, v, for $d=0.05$, $\tau=1$, $\nu=10^{-3}$, and $\alpha=0$. Incidentally, $v=-1$, 0, and τ correspond to the island propagating with the local electron, $\mathbf{E}\times\mathbf{B}$, and ion fluids, respectively. The dashed curve is the standard result (51).

$$Y(X,\theta) = \text{Re}\sum_{m=0}^{\infty} c_m(X)e^{i[k_X(X)X+m\theta]} = \text{Re}\sum_{m=0}^{\infty} d_m(X)e^{im\theta}. \tag{55}$$

Here, the sign of k_X should be chosen such that $d(k_X)/dv>0$ as $X\to\infty$ (i.e., the asymptotic group velocity of the emitted waves should be positive). However, this is automatically taken care of by the small complex term in Eq. (53).

It follows from Eqs. (54) and (55) that the physical solution can be written

$$Y(X,\theta) = Y_c(X,\theta) + Y_s(X,\theta), \tag{56}$$

where

$$Y_c(X,\theta) = \text{Re}[Y^{(c)}(X,\theta)] \tag{57}$$

has the symmetry of $\cos\theta$, whereas

$$Y_s(X,\theta) = -\sum_{m=1}^{\infty} D_m(X)\sin(m\theta) \tag{58}$$

has the symmetry of $\sin\theta$. Here, the D_m are real, and are given by

$$D_m(X) = \frac{2}{\pi}\int_0^{\pi} \text{Im}[Y^{(c)}(X,\theta)]\cos(m\theta)d\theta. \tag{59}$$

The fact that when an island is emitting waves there is a nonzero component of $Y(X,\theta)$ with the symmetry of $\sin\theta$ implies that, in this case, there is a net electromagnetic force acting on the island region [see Eqs. (32) and (48)]. This force arises as a reaction to the momentum carried off by the radiated waves.[8] An equal and opposite force will develop in any region of the plasma where the waves are absorbed. We expect this region to be located around the ion Landau resonance. In this paper, it is implicitly assumed that the absorption region is situated relatively far from the island.

III. NUMERICAL RESULTS

The mathematical formalism described in the previous section has been implemented in a finite-difference code.

Figure 1 shows the stability parameter, $\mathcal{C}$, and the force parameter, $\mathcal{D}$, calculated as functions of the island phase-velocity parameter, v, in the absence of ion compressibility

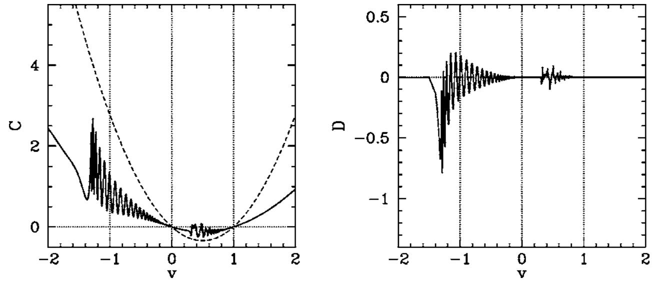

FIG. 2. The stability parameter, $\mathcal{C}$, and the force parameter, $\mathcal{D}$, calculated as functions of the island phase-velocity parameter, v, for $d=0.05$, $\tau=1$, $\nu=10^{-3}$, and $\alpha=0.5$. Incidentally, $v=-1$, 0, and τ correspond to the island propagating with the local electron, $\mathbf{E}\times\mathbf{B}$, and ion fluids, respectively.

(i.e., $\alpha=0$). It can be seen that the $\mathcal{C}$ curve has the general form predicted in Eq. (51) (i.e., it is essentially parabolic, being negative for $0<v<\tau$ and positive otherwise), except that the constant in front is obviously somewhat less than 1.38. In fact, it turns out that the numerically determined constant only approaches 1.38 in the limit in which the drift-wave parameter, d, becomes extremely small. Hence, we conclude that, in general, the standard result Eq. (51) somewhat overestimates the stabilizing or destabilizing effect of the ion polarization term in the Rutherford island width evolution equation.

In the absence of ion compressibility, drift waves propagate when $-1<v<\tau$. It can be seen, from Fig. 1, that in this range of island phase velocities the stability parameter curve exhibits rapid oscillations (similar to those reported in Ref. 8) that are superimposed on its underlying parabolic form. These oscillations are associated with drift-wave emission by the magnetic island, and are much stronger when the island is propagating in the electron, rather than the ion, diamagnetic direction (in the local $\mathbf{E}\times\mathbf{B}$ frame). The reason for this is that $\omega=\omega_{*e}$ is a cutoff for drift-wave propagation, so that close to this frequency the waves have long wavelengths and are weakly damped. By contrast, $\omega=0$ and $\omega=\omega_{*i}$ are resonances, so that between these two frequencies, drift waves have short wavelengths and are heavily damped. It can also be seen that, as expected, the force parameter, $\mathcal{D}$, is zero, except in the range of phase velocities in which the island is strongly radiating.

Figures 2 and 3 show the stability and force parameters calculated as functions of the island phase-velocity parameter in the presence of ion compressibility (i.e., $\alpha>0$). The stability and force parameter curves are very similar to those shown in Fig. 1. The main difference is that, in the presence of ion compressibility, the range of phase velocities in which the island radiates waves is extended in the electron diamagnetic direction. However, there is no evidence for any overall stabilizing effect of ion compressibility, as reported in Ref. 9.

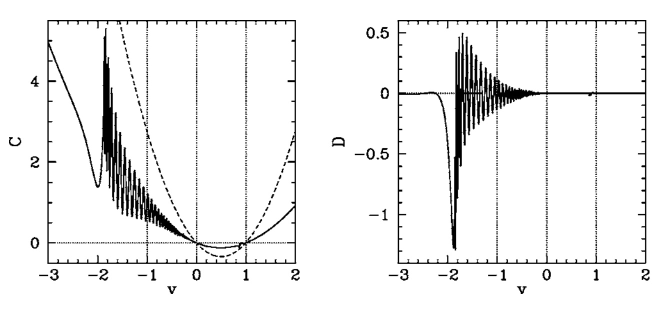

FIG. 3. The stability parameter, $\mathcal{C}$, and the force parameter, $\mathcal{D}$, calculated as functions of the island phase-velocity parameter, v, for $d=0.05$, $\tau=1$, $\nu=10^{-3}$, and $\alpha=1.0$. Incidentally, $v=-1$, 0, and τ correspond to the island propagating with the local electron, $\mathbf{E}\times\mathbf{B}$, and ion fluids, respectively.

IV. SUMMARY

We have developed a mathematical formalism for calculating the ion polarization term in the Rutherford island width evolution equation, as well as the net electromagnetic force acting on the island region, in the presence of drift-acoustic waves. This formalism is appropriate for relatively large (i.e., $W\gg d_\beta$), isolated, constant-ψ, magnetic islands, in slab geometry.

In the absence of ion compressibility, the ion polarization term is found to be similar in form to the standard analytic prediction [see Eqs. (31) and (51)], but somewhat smaller in magnitude. The magnetic island is found to radiate drift waves when its phase velocity lies between that of the local ion and electron fluid velocities. However, the drift-wave emission is far stronger when the island propagates in the electron, rather than the ion, direction (in the local $\mathbf{E}\times\mathbf{B}$ frame). Strong drift-wave emission gives rise to rapid oscillations in the ion polarization term (as a function of the island phase velocity). Moreover, the momentum carried off by the drift waves leads to a net electromagnetic force acting on the island region. This force also oscillates rapidly as the island phase velocity varies.

When parallel ion compressibility is taken into account, the drift waves become drift-acoustic waves. However, the situation remains very similar to that described above, except that the range of phase velocities in which the island radiates waves is extended in the electron diamagnetic direction. However, there is no evidence for an overall stabilizing effect of ion compressibility, as claimed in Ref. 9.

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APPENDIX A: ALTERNATIVE NONLINEAR ANALYSIS

In the following, we set $\tau=0$ for the sake of simplicity. Equation (15) can be integrated to give

$$\phi - Z = F(\psi). \tag{A1}$$

The above equation can be combined with Eq. (18) to give

$$V_z + \psi = G(\phi). \tag{A2}$$

Equations (16) and (17), and the previous two equations, yield

$$d_\beta^2 \nabla^2 \phi + F(\psi) - c_\beta^2 G'(\phi)\psi = H(\phi) \tag{A3}$$

and

$$\tilde{J} = [F'(\psi)\tilde{\phi} - c_\beta^2 \widetilde{G(\phi)}]/d_\beta^2. \tag{A4}$$

Here, $'$ denotes a derivative, and F, G, and H are, as yet, unspecified functions.

Outside the island separatrix, making the expansion (19)–(21), the above equations give

$$F(\psi) = \phi_0(\psi) - Z_0(\psi), \tag{A5}$$

$$G(\phi_0) = V_0(\psi) + \psi, \tag{A6}$$

$$H(\phi_0) = F(\psi) - c_\beta^2 G'(\phi_0)\psi, \tag{A7}$$

to lowest order in d_β^2. Note that the functions F, G, and H have now been completely specified in terms of ψ and the profile functions ϕ_0, Z_0, and V_0. To next order, we obtain $\phi_1 = Z_1 = M V_1/N$ and

$$\tilde{J} = M(\psi)K(\phi_0)\tilde{\phi}_1, \tag{A8}$$

plus

$$d_\beta^2 \nabla^2 \phi_1 - K(\phi_0)\phi_1 = -\nabla^2 \phi_0, \tag{A9}$$

where

$$K(\phi_0) = H'(\phi_0) + c_\beta^2 G''(\phi_0)\psi = \frac{F'(\psi) - c_\beta^2 G'(\phi_0)}{M(\psi)}$$
$$= 1 - L(\psi)/M(\psi) - c_\beta^2 N(\psi)/M^2(\psi). \tag{A10}$$

Here, $M(\psi) = d\phi_0/d\psi$, $L(\psi) = dZ_0/d\psi$, and $N(\psi) = dV_0/d\psi + 1$. Equations (A8) and (A9) are essentially identical to Eqs. (26) and (27) (with $\tau = 0$), respectively.

APPENDIX B: LINEAR ANALYSIS

We start from the dissipationless form of Eqs. (4)–(7),

$$\frac{\partial \psi}{\partial t} = [\phi, \psi] - [Z, \psi], \tag{B1}$$

$$\frac{\partial Z}{\partial t} = [\phi, Z] + c_\beta^2 [V_z, \psi] + d_\beta^2 [J, \psi] - \alpha Z, \tag{B2}$$

$$\frac{\partial \nabla^2 \phi}{\partial t} = [\phi, \nabla^2 \phi] - \frac{\tau}{2}(\nabla^2[\phi, Z] + [\nabla^2 \phi, Z] + [\nabla^2 Z, \phi])$$
$$+ [J, \psi], \tag{B3}$$

$$\frac{\partial V_z}{\partial t} = [\phi, V_z] + [Z, \psi]. \tag{B4}$$

We have, however, added a small term to Eq. (B2) that is designed to damp drift-acoustic waves. Here, $0 < \alpha \ll 1$. Let

$$\psi(x,y,t) = -\frac{1}{2}x^2, \tag{B5}$$

$$Z(x,y,t) = -[V_*/(1+\tau)]x + \tilde{Z}(x)e^{i(ky-\omega t)}, \tag{B6}$$

$$\phi(x,y,t) = -V_{EB}x + \tilde{\phi}(x)e^{i(ky-\omega t)}, \tag{B7}$$

$$V_z(x,y,t) = \tilde{V}_z(x)e^{i(ky-\omega t)}, \tag{B8}$$

$$J(x,y,t) = \tilde{J}(x)e^{i(ky-\omega t)}, \tag{B9}$$

where ~ denotes a perturbed quantity. Here, V_{EB} is the local $\mathbf{E}\times\mathbf{B}$ velocity, and $V_* > 0$ is the local diamagnetic velocity. Of course, the phase velocity of the perturbation is $V = \omega/k$.

Substituting (B5)–(B9) into Eqs. (B1)–(B4), and retaining only first-order terms, we obtain

$$d^2 \frac{d^2\tilde{\phi}}{dX^2} - \left[\frac{v + 1 - (\alpha^2/v)X^2 + i\nu}{v - \tau}\right]\tilde{\phi} \simeq 0. \tag{B10}$$

Here, $X = x/w$, $d = d_\beta/w$, $\alpha = c_\beta w(1+\tau)/V_*$, $v = (V - V_{EB})(1+\tau)/V_*$, and $\nu = \alpha(1+\tau)/(kV_*) > 0$. The small imaginary term in the above equation damps drift-acoustic waves. Note that, in the limit $X \gg 1$, the above wave equation becomes identical in form to the homogeneous part of Eq. (53). We, therefore, expect the small imaginary term added to Eq. (53) to cause the physical outgoing wave solution to decay as $X \to \infty$. Conversely, the added term will cause the unphysical incoming wave solution to blow up as $X \to \infty$.

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