

# Cylindrical Tearing Mode Analysis

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## 1 Normalization

All lengths are normalized to  $R_0$ . All magnetic field-strengths are normalized to  $B_0$ . All current densities are normalized to  $B_0/(R_0 \mu_0)$ .

## 2 Coordinates

Adopt cylindrical coordinates  $r, \theta, z$ .

## 3 Plasma Equilibrium

The equilibrium magnetic field is written

$$\mathbf{B} = \nabla\Psi(r) \times \mathbf{e}_z + \mathbf{e}_z. \quad (1)$$

So,

$$B_r = 0, \quad (2)$$

$$B_\theta = -\frac{d\Psi}{dr}, \quad (3)$$

$$B_z = 1. \quad (4)$$

The safety-factor profile is

$$q(r) = \frac{r B_z}{B_\theta} = \frac{r}{B_\theta}. \quad (5)$$

The equilibrium current density is written

$$\mathbf{J} = J_z(r) \mathbf{e}_z, \quad (6)$$

where

$$J_z = \frac{1}{r} \frac{d(r B_\theta)}{dr} = \frac{1}{r} \frac{d}{dr} \left( \frac{r^2}{q} \right) = \frac{2-s}{q}, \quad (7)$$

and

$$s(r) = \frac{d \ln q}{dr}. \quad (8)$$

## 4 Cylindrical Tearing Mode Equation

Let

$$\mathbf{b} = \nabla [\psi(r) e^{i(m\theta - nz - \omega t)}] \times \mathbf{e}_z \quad (9)$$

$$\mathbf{j} \simeq - \left[ \frac{1}{r} \frac{d}{dr} \left( r \frac{d\psi}{dr} \right) - \frac{m^2}{r^2} \psi \right] e^{i(m\theta - nz - \omega t)} \mathbf{e}_z, \quad (10)$$

be the perturbed magnetic field and current density, respectively. Here, we have assumed that  $r \ll 1$ . Linearized force balance yields

$$(\mathbf{J} \cdot \nabla) \mathbf{b} + (\mathbf{j} \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{j} - (\mathbf{b} \cdot \nabla) \mathbf{J} = 0. \quad (11)$$

The  $z$  component of the previous equation gives the cylindrical tearing mode equation:

$$\frac{d^2\psi}{dr^2} + \frac{1}{r} \frac{d\psi}{dr} - \frac{m^2}{r^2} \psi - \frac{J'_z \psi}{r(1/q - n/m)} = 0, \quad (12)$$

where  $' \equiv d/dr$ . In the following,  $q(r)$  and  $J_z(r)$  are taken from the EQDSK equilibrium, rather than being related according to Eq. (7).

## 5 Behavior in Vicinity of Rational Surface

Suppose that  $q(r_s) = m/n$ . Let  $\rho = r - r_s$ . Expanding the previous equation about  $r = r_s$ , we obtain

$$\frac{d^2\psi}{d\rho^2} + \frac{1}{r_s} \left( 1 - \frac{\rho}{r_s} \right) \frac{d\psi}{d\rho} - \frac{m^2}{r_s^2} \left( 1 - \frac{2\rho}{r_s} \right) \psi - \left( \frac{\alpha}{\rho} + \beta \right) \psi = 0, \quad (13)$$

where

$$\alpha = - \left( \frac{q J'_z}{s} \right)_{r=r_s}, \quad (14)$$

$$\beta = \alpha \left( \frac{J''_z}{J'_z} + \frac{s-1}{r_s} - \frac{q''}{2q'} \right)_{r=r_s}. \quad (15)$$

We obtain

$$\begin{aligned} \psi(\rho) = & \Psi \left[ 1 + \left( \frac{\Sigma' \pm \Delta'}{2} \right) \rho \right. \\ & + \left\{ \alpha \left( \frac{\Sigma' \pm \Delta'}{4} \right) \left( 1 - \frac{1}{\alpha r_s} \right) + \frac{1}{2} \left[ \frac{m^2}{r_s^2} + \beta - \frac{\alpha^2}{2} \left( 3 - \frac{1}{\alpha r_s} \right) \right] \right\} \rho^2 \\ & \left. + \alpha \left\{ \rho + \frac{\alpha}{2} \left( 1 - \frac{1}{\alpha r_s} \right) \rho^2 \right\} \ln |\rho| + \mathcal{O}(\rho^3) \right]. \quad (16) \end{aligned}$$

Here, the plus sign corresponds to  $\rho > 0$ , whereas the minus sign corresponds to  $\rho < 0$ . It follows that

$$\begin{aligned} \psi'(\rho) = \Psi & \left[ \frac{\Sigma' \pm \Delta'}{2} + \alpha \right. \\ & + \left\{ \alpha \left( \frac{\Sigma' \pm \Delta'}{2} \right) \left( 1 - \frac{1}{\alpha r_s} \right) + \frac{m^2}{r_s^2} + \beta - \frac{\alpha^2}{2} \left( 3 - \frac{1}{\alpha r_s} \right) + \frac{\alpha^2}{2} \left( 1 - \frac{1}{\alpha r_s} \right) \right\} \rho \\ & \left. + \alpha \ln |\rho| + \alpha^2 \left( 1 - \frac{1}{\alpha r_s} \right) \rho \ln |\rho| + \mathcal{O}(\rho^2) \right]. \end{aligned} \quad (17)$$

Now,

$$\lambda_+ \equiv \left. \frac{\psi'}{\psi} \right|_{\rho=\delta} = \frac{a_{1+} + b_{1+} (\Sigma' + \Delta')}{a_{2+} + b_{2+} (\Sigma' + \Delta')}, \quad (18)$$

where

$$a_{1+} = \alpha (1 + \ln \delta) + \left\{ \frac{m^2}{r_s^2} + \beta - \frac{\alpha^2}{2} \left( 3 - \frac{1}{\alpha r_s} \right) + \alpha^2 \left( 1 - \frac{1}{\alpha r_s} \right) \left( \frac{1}{2} + \ln \delta \right) \right\} \delta, \quad (19)$$

$$b_{1+} = \frac{1}{2} + \frac{\alpha}{2} \left( 2 - \frac{1}{\alpha r_s} \right) \delta, \quad (20)$$

$$a_{2+} = 1 + \alpha \delta \ln \delta, \quad (21)$$

$$b_{2+} = \frac{\delta}{2}. \quad (22)$$

It follows that

$$\Sigma' + \Delta' = \frac{a_{2+} \lambda_+ - a_{1+}}{b_{1+} - b_{2+} \lambda_+}. \quad (23)$$

Likewise,

$$\lambda_- \equiv \left. \frac{\psi'}{\psi} \right|_{\rho=-\delta} = \frac{a_{1-} + b_{1-} (\Sigma' - \Delta')}{a_{2-} + b_{2-} (\Sigma' - \Delta')}, \quad (24)$$

where

$$a_{1-} = \alpha (1 + \ln \delta) - \left\{ \frac{m^2}{r_s^2} + \beta - \frac{\alpha^2}{2} \left( 3 - \frac{1}{\alpha r_s} \right) + \alpha^2 \left( 1 - \frac{1}{\alpha r_s} \right) \left( \frac{1}{2} + \ln \delta \right) \right\} \delta, \quad (25)$$

$$b_{1-} = \frac{1}{2} - \frac{\alpha}{2} \left( 2 - \frac{1}{\alpha r_s} \right) \delta, \quad (26)$$

$$a_{2-} = 1 - \alpha \delta \ln \delta, \quad (27)$$

$$b_{2-} = -\frac{\delta}{2}. \quad (28)$$

It follows that

$$\Sigma' - \Delta' = \frac{a_{2-} \lambda_- - a_{1-}}{b_{1-} - b_{2-} \lambda_-}. \quad (29)$$

Hence,

$$\Sigma' = \frac{1}{2} \left( \frac{a_{2+} \lambda_+ - a_{1+}}{b_{1+} - b_{2+} \lambda_+} + \frac{a_{2-} \lambda_- - a_{1-}}{b_{1-} - b_{2-} \lambda_-} \right), \quad (30)$$

$$\Delta' = \frac{1}{2} \left( \frac{a_{2+} \lambda_+ - a_{1+}}{b_{1+} - b_{2+} \lambda_+} - \frac{a_{2-} \lambda_- - a_{1-}}{b_{1-} - b_{2-} \lambda_-} \right). \quad (31)$$

## 6 Solution Launched from Magnetic Axis

Let

$$\psi(r) = r^m f(r). \quad (32)$$

It follows that

$$\psi'(r) = m r^{m-1} f + r^m f'. \quad (33)$$

The cylindrical tearing mode equation transforms into

$$f'' + (1 + 2m) \frac{f'}{r} - \frac{J'_z f}{r(1/q - n/m)} = 0. \quad (34)$$

The well-behaved solution launched from the magnetic axis is such that

$$f(r) = 1 + \left[ \frac{c}{4(1+m)} \right] r^2, \quad (35)$$

where

$$c = \left( \frac{J''_z}{1/q - n/m} \right)_{r=0}. \quad (36)$$

## 7 Solution Launched from Plasma Boundary

Suppose that the plasma boundary lies at  $r = a$ . Let

$$\psi(r) = r^{-m} g(r). \quad (37)$$

It follows that

$$\psi'(r) = -m r^{-m-1} g + r^{-m} g'. \quad (38)$$

The cylindrical tearing mode equation transforms into

$$g'' + (1 - 2m) \frac{g'}{r} - \frac{J'_z g}{r(1/q - n/m)} = 0. \quad (39)$$

In the absence of a wall, or external currents flowing in non-axisymmetric field-coils, the well-behaved solution launched from the plasma boundary is such that

$$g(a) = 1, \quad (40)$$

$$g'(a) = 0. \quad (41)$$

## 8 Resistive Wall

Suppose that a thin resistive wall of radius  $r_w > a$ , uniform thickness  $\delta_w$ , and uniform resistivity  $\eta_w$ , is located outside the plasma. We can now launch two different solutions from the plasma boundary. The *no wall* solution is such that

$$g_{nw}(a) = 1, \quad (42)$$

$$g'_{nw}(a) = 0. \quad (43)$$

Let the associated  $r_s \Sigma'$  and  $r_s \Delta'$  values be denoted  $\Sigma^{nw}$  and  $\Delta^{nw}$ , respectively. The *perfect wall* solution is such that

$$g_{pw}(a) = 1, \quad (44)$$

$$g'_{pw}(a) = -\frac{2m/a}{(r_w/a)^{2m} - 1}. \quad (45)$$

Let the associated  $r_s \Sigma'$  and  $r_s \Delta'$  values be denoted  $\Sigma^{pw}$  and  $\Delta^{pw}$ , respectively.

Asymptotic matching in the presence of the wall yields

$$\Delta\Psi_s = \Delta^{pw} \Psi_s + \Sigma_w \Psi_w, \quad (46)$$

$$\Delta\Psi_w = \Delta_w \Psi_w + \Sigma_w \Psi_s. \quad (47)$$

Here,  $\Psi_s \equiv \psi(r_s)$ ,  $\Psi_w \equiv \psi(r_w)$ , and

$$\Delta\Psi_s \equiv \left[ r \frac{d\psi}{dr} \right]_{r_s-}^{r_s+}, \quad (48)$$

$$\Delta\Psi_w \equiv \left[ r \frac{d\psi}{dr} \right]_{r_w-}^{r_w+}. \quad (49)$$

We can make the identifications

$$\Delta^{nw} = \frac{\Delta\Psi_s}{\Psi_s} \Big|_{\Delta\Psi_w=0} = \Delta^{pw} - \frac{\Sigma_w^2}{\Delta_w}, \quad (50)$$

$$\Sigma_w = \frac{2m (r_s/r_w)^m}{[1 - (a/r_w)^{2m}] g_{pw}(r_s)}. \quad (51)$$

Hence,

$$\Delta_w = - \left( \frac{\Sigma_w^2}{\Delta^{nw} - \Delta^{pw}} \right). \quad (52)$$

Standard analysis reveals that

$$\Delta\Psi_w = \tau_w \frac{d\Psi_w}{dt}, \quad (53)$$

where

$$\tau_w = \frac{\mu_0 \delta_w r_w}{\eta_w}. \quad (54)$$

It follows that

$$\Delta \Psi_s = \Delta^{pw} \Psi_s + \Sigma_w \Psi_w, \quad (55)$$

$$\tau_w \frac{d\Psi_w}{dt} = \Delta_w \Psi_w + \Sigma_w \Psi_s. \quad (56)$$