Program PHASE

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1 Plasma Angular Velocity Evolution

1.1 Plasma Angular Equations of Motion

We can write

$$\varOmega_\theta(r,t) = \varOmega_{\theta\,0}(r) + \varDelta\varOmega_\theta(r,t), \tag{1}$$

$$\varOmega_\phi(r,t) = \varOmega_{\phi\,0}(r) + \varDelta\varOmega_\phi(r,t), \tag{2}$$

where $\varOmega_\theta(r,t)$ and $\varOmega_\phi(r,t)$ are the poloidal and toroidal plasma angular velocity profiles, respectively, whereas $\varOmega_{\theta\,0}(r)$ and $\varOmega_{\phi\,0}(r)$ are the corresponding unperturbed profiles, and $\varDelta\varOmega_\theta(r,t)$ and $\varDelta\varOmega_\phi(r,t)$ are the respective modifications to the profiles induced by the electromagnetic torques exerted at the various resonant surfaces within the plasma. The modifications to the angular velocity profiles are governed by the poloidal and toroidal angular equations of motion of the plasma, which take the respective forms

$$4\pi^2\,R_0\left[(1+2\,Q^2)\,\rho\,r^3\,\frac{\partial\varDelta\varOmega_\theta}{\partial t} - \frac{\partial}{\partial r}\left(\mu\,r^3\,\frac{\partial\varDelta\varOmega_\theta}{\partial r}\right) + \frac{\rho}{\tau_\theta}\,r^3\,\varDelta\varOmega_\theta + \frac{\rho}{\tau_{\rm cx}}\,r^3\,\varDelta\varOmega_\theta\right]$$
$$= \sum_{k=1,K}\delta T_{\theta\,k}\,\delta(r-r_k), \tag{3}$$

$$4\pi^2\,R_0^3\left[\rho\,r\,\frac{\partial\varDelta\varOmega_\phi}{\partial t} - \frac{\partial}{\partial r}\left(\mu\,r\,\frac{\partial\varDelta\varOmega_\phi}{\partial r}\right) + \frac{\rho}{\tau_{\rm cx}}\,r\,\varDelta\varOmega_\phi\right] = \sum_{k=1,K}\delta T_{\phi\,k}\,\delta(r-r_k), \tag{4}$$

and are subject to the spatial boundary conditions

$$\frac{\partial\varDelta\varOmega_\theta(0,t)}{\partial r} = \frac{\partial\varDelta\varOmega_\phi(0,t)}{\partial r} = 0, \tag{5}$$

$$\varDelta\varOmega_\theta(a,t) = \varDelta\varOmega_\phi(a,t) = 0. \tag{6}$$

Here, $\mu(r)$ is the anomalous plasma perpendicular ion viscosity (due to plasma turbulence), whereas $\rho(r)$ is the plasma mass density profile. Furthermore, $1/\tau_\theta(r)$ is the neoclassical poloidal flow-damping rate. Finally, $1/\tau_{\rm cx}(r) = \langle n_n\rangle\,\langle \sigma\, v\rangle_i^{\rm cx}$ is the charge-exchange flow-damping rate.

According to standard neoclassical theory,

$$\frac{1}{\tau_\theta(r)} = \left(\frac{I^{\,2}}{\langle|\nabla\psi|^{\,2}\rangle}\right)\,\frac{\mu_{00}^{\,i}}{\tau_{ii}}. \tag{7}$$

1.2 Simplified Plasma Angular Equations of Motion

It is convenient to write

$$\Delta\Omega_\theta(r,t) = \sum_{k=1,K}\Delta\Omega_{\theta\,k}(r,t), \tag{8}$$

$$\Delta\Omega_\phi(r,t) = \sum_{k=1,K}\Delta\Omega_{\phi\,k}(r,t), \tag{9}$$

where

$$4\pi^{\,2}\,R_0\left[(1+2\,Q^{\,2})\,\rho\,r^{\,3}\,\frac{\partial\Delta\Omega_{\theta\,k}}{\partial t} - \frac{\partial}{\partial r}\left(\mu\,r^{\,3}\,\frac{\partial\Delta\Omega_{\theta\,k}}{\partial r}\right) + \frac{\rho}{\tau_\theta}\,r^{\,3}\,\Delta\Omega_{\theta\,k} + \frac{\rho}{\tau_{\rm cx}}\,r^{\,3}\,\Delta\Omega_{\theta\,k}\right] = \delta T_{\theta\,k}\,\delta(r-r_k), \tag{10}$$

$$4\pi^{\,2}\,R_0^{3}\left[\rho\,r\,\frac{\partial\Delta\Omega_{\phi\,k}}{\partial t} - \frac{\partial}{\partial r}\left(\mu\,r\,\frac{\partial\Delta\Omega_{\phi\,k}}{\partial r}\right) + \frac{\rho}{\tau_\phi}\,r\,\Delta\Omega_{\phi\,k}\right] = \delta T_{\phi\,k}\,\delta(r-r_k), \tag{11}$$

and

$$\frac{\partial\Delta\Omega_{\theta\,k}(0,t)}{\partial r} = \frac{\partial\Delta\Omega_{\phi\,k}(0,t)}{\partial r} = 0, \tag{12}$$

$$\Delta\Omega_{\theta\,k}(a,t) = \Delta\Omega_{\phi\,k}(a,t) = 0. \tag{13}$$

In the presence of poloidal and toroidal flow damping, the modified angular velocity profiles, $\Delta\Omega_{\theta\,k}$ and $\Delta\Omega_{\phi\,k}$, are radially localized in the vicinity of the kth resonant surface. Hence, it is a good approximation to write Eqs. (10) and (11) in the simplified forms

$$4\pi^{\,2}\,R_0\left[(1+2\,Q_k^{2})\,\rho_k\,r^{\,3}\,\frac{\partial\Delta\Omega_{\theta\,k}}{\partial t} - \mu_k\,\frac{\partial}{\partial r}\left(r^{\,3}\,\frac{\partial\Delta\Omega_{\theta\,k}}{\partial r}\right) + \frac{\rho_k}{\tau_{\theta\,k}}\,r^{\,3}\,\Delta\Omega_{\theta\,k} + \frac{\rho_k}{\tau_{{\rm cx}\,k}}\,r^{\,3}\,\Delta\Omega_{\theta\,k}\right] = \delta T_{\theta\,k}\,\delta(r-r_k), \tag{14}$$

$$4\pi^{\,2}\,R_0^{3}\left[\rho_k\,r\,\frac{\partial\Delta\Omega_{\phi\,k}}{\partial t} - \mu_k\frac{\partial}{\partial r}\left(r\,\frac{\partial\Delta\Omega_{\phi\,k}}{\partial r}\right) + \frac{\rho_k}{\tau_{{\rm cx}\,k}}\,r\,\Delta\Omega_{\phi\,k}\right] = \delta T_{\phi\,k}\,\delta(r-r_k), \tag{15}$$

where $Q_k = Q(r_k)$, $\rho_k = \rho(r_k)$, $\mu_k = \mu(r_k)$, $\tau_{\theta\,k} = \tau_\theta(r_k)$, and $\tau_{{\rm cx}\,k} = \tau_{\rm cx}(r_k)$.

1.3 Normalized Plasma Equations of Angular Motion

Let

$$\tau_A = \left(\frac{\mu_0\,\rho_0\,a^{\,2}}{B_0^{\,2}}\right)^{1/2}, \tag{16}$$

where $\rho_0 = \rho(0)$, and

$$\rho(r) \simeq m_i\,[n_i(r) + n_b(r)] + m_I\,n_I(r) \tag{17}$$

is the plasma mass density. Furthermore, let $\hat{r} = r/a$, $\hat{r}_k = r_k/a$, $\hat{a} = a/R_0$, $\hat{t} = t/\tau_A$, $\hat{\rho}_k = \rho_k/\rho_0$, $\tau_{M\,k} = \rho_k\,a^2/\mu_k = a^2/\chi_\phi(r_k)$, $\hat{\tau}_{M\,k} = \tau_{M\,k}/\tau_A$, $\hat{\tau}_{\theta\,k} = \tau_{\theta\,k}/\tau_A$, $\hat{\tau}_{{\rm cx}\,k} = \tau_{{\rm cx}\,k}/\tau_A$, $\hat{\varOmega}_\theta = \tau_A\,\varOmega_\theta$, $\hat{\varOmega}_{\theta\,0} = \tau_A\,\varOmega_{\theta\,0}$, $\varDelta\hat{\varOmega}_{\theta\,k} = \tau_A\,\varDelta\varOmega_{\theta\,k}$, $\hat{\varOmega}_\phi = \tau_A\,\varOmega_\phi$, $\hat{\varOmega}_{\phi\,0} = \tau_A\,\varOmega_{\phi\,0}$, and $\varDelta\hat{\varOmega}_{\phi\,k} = \tau_A\,\varDelta\varOmega_{\phi\,k}$. It follows that

$$\hat{\varOmega}_\theta(\hat{r},\hat{t}) = \hat{\varOmega}_{\theta\,0}(\hat{r}) + \sum_{k=1,K}\varDelta\hat{\varOmega}_{\theta\,k}(\hat{r},\hat{t}), \tag{18}$$

$$\hat{\varOmega}_\phi(\hat{r},\hat{t}) = \hat{\varOmega}_{\phi\,0}(\hat{r}) + \sum_{k=1,K}\varDelta\hat{\varOmega}_{\phi\,k}(\hat{r},\hat{t}), \tag{19}$$

where

$$(1+2\,Q_k^2)\,\hat{r}^{\,3}\,\frac{\partial\varDelta\hat{\varOmega}_{\theta\,k}}{\partial\hat{t}} - \frac{1}{\hat{\tau}_{M\,k}}\,\frac{\partial}{\partial\hat{r}}\!\left(\hat{r}^{\,3}\,\frac{\partial\varDelta\hat{\varOmega}_{\theta\,k}}{\partial\hat{r}}\right) + \frac{1}{\hat{\tau}_{\theta\,k}}\,\hat{r}^{\,3}\,\varDelta\hat{\varOmega}_{\theta\,k} + \frac{1}{\hat{\tau}_{{\rm cx}\,k}}\,\hat{r}^{\,3}\,\varDelta\hat{\varOmega}_{\theta\,k} = -\frac{m_k}{2\,\hat{\rho}_k\,\hat{a}^{\,2}}\,\delta\hat{T}_k\,\delta(\hat{r}-\hat{r}_k), \tag{20}$$

$$\hat{r}\,\frac{\partial\varDelta\hat{\varOmega}_{\phi\,k}}{\partial\hat{t}} - \frac{1}{\hat{\tau}_{M\,k}}\,\frac{\partial}{\partial\hat{r}}\!\left(\hat{r}\,\frac{\partial\varDelta\hat{\varOmega}_{\phi\,k}}{\partial\hat{r}}\right) + \frac{1}{\hat{\tau}_{{\rm cx}\,k}}\,\hat{r}\,\varDelta\hat{\varOmega}_{\phi\,k} = \frac{n}{2\,\hat{\rho}_k}\,\delta\hat{T}_k\,\delta(\hat{r}-\hat{r}_k), \tag{21}$$

and

$$\frac{\partial\varDelta\hat{\varOmega}_{\theta\,k}(0,\hat{t})}{\partial\hat{r}} = \frac{\partial\varDelta\hat{\varOmega}_{\phi\,k}(0,\hat{t})}{\partial\hat{r}} = 0, \tag{22}$$

$$\varDelta\hat{\varOmega}_{\theta\,k}(1,\hat{t}) = \varDelta\hat{\varOmega}_{\phi\,k}(1,\hat{t}) = 0. \tag{23}$$

1.4 Solution of Plasma Angular Equations of Motion

Let

$$\varDelta\hat{\varOmega}_{\theta\,k}(\hat{r},\hat{t}) = -\frac{1}{m_k}\sum_{p=1,P}\alpha_{k,p}(\hat{t})\,\frac{y_p(\hat{r})}{y_p(\hat{r}_k)}, \tag{24}$$

$$\varDelta\hat{\varOmega}_{\phi\,k}(\hat{r},\hat{t}) = \frac{1}{n}\sum_{p=1,P}\beta_{k,p}(\hat{t})\,\frac{z_p(\hat{r})}{z_p(\hat{r}_k)}, \tag{25}$$

where

$$y_p(\hat{r}) = \frac{J_1(j_{1,p}\,\hat{r})}{\hat{r}}, \tag{26}$$

$$z_p(\hat{r}) = J_0(j_{0,p}\,\hat{r}), \tag{27}$$

and $P \gg 1$. Here, $J_m(z)$ is a standard Bessel function, and $j_{m,p}$ denotes the pth zero of the $J_m(z)$ Bessel function. Note that Eqs. (24) and (25) automatically satisfy the boundary conditions Eqs. (22) and (23).

It is easily demonstrated that

$$\frac{d}{d\hat{r}}\left(\hat{r}^{\,3}\,\frac{dy_p}{d\hat{r}}\right) = -j_{1,p}^{\,2}\,\hat{r}^{\,3}\,y_p, \tag{28}$$

$$\frac{d}{d\hat{r}}\left(\hat{r}\,\frac{dz_p}{d\hat{r}}\right) = -j_{0,p}^{\,2}\,\hat{r}\,z_p, \tag{29}$$

and

$$\int_0^1 \hat{r}^{\,3}\,y_p(\hat{r})\,y_q(\hat{r})\,d\hat{r} = \frac{1}{2}\,\left[J_2(j_{1,p})\right]^2\,\delta_{pq}, \tag{30}$$

$$\int_0^1 \hat{r}\,z_p(\hat{r})\,z_q(\hat{r})\,d\hat{r} = \frac{1}{2}\,\left[J_1(j_{0,p})\right]^2\,\delta_{pq} \tag{31}$$

Hence, Eqs. (20) and (21) yield

$$(1+2\,Q_k^2)\,\frac{d\alpha_{k,p}}{d\hat{t}} + \left(\frac{j_{1,p}^{\,2}}{\hat{\tau}_{M\,k}} + \frac{1}{\hat{\tau}_{\theta\,k}} + \frac{1}{\hat{\tau}_{{\rm cx}\,k}}\right)\alpha_{k,p} = \frac{m_k^2\,[y_p(\hat{r}_k)]^{\,2}}{\hat{\rho}_k\,\hat{a}^{\,2}\,[J_2(j_{1,p})]^{\,2}}\,\delta\hat{T}_k, \tag{32}$$

$$\frac{d\beta_{k,p}}{d\hat{t}} + \left(\frac{j_{0,p}^{\,2}}{\hat{\tau}_{M\,k}} + \frac{1}{\hat{\tau}_{{\rm cx}\,k}}\right)\beta_{k,n} = \frac{n^{\,2}\,[z_p(\hat{r}_k)]^{\,2}}{\hat{\rho}_k\,[J_1(j_{0,p})]^{\,2}}\,\delta\hat{T}_k. \tag{33}$$

It follows that

$$\hat{\Omega}_\theta(\hat{r}_k,\hat{t}) = \hat{\Omega}_{\theta\,0}(\hat{r}_k) - \sum_{k'=1,K}^{p=1,P}\frac{\alpha_{k',p}(\hat{t})}{m_{k'}}\,\frac{y_p(\hat{r}_k)}{y_p(\hat{r}_{k'})}, \tag{34}$$

$$\hat{\Omega}_\phi(\hat{r}_k,\hat{t}) = \hat{\Omega}_{\phi\,0}(\hat{r}_k) + \sum_{k'=1,K}^{p=1,P}\frac{\beta_{k',p}(\hat{t})}{n}\,\frac{z_p(\hat{r}_k)}{z_p(\hat{r}_{k'})}. \tag{35}$$

Here, k indexes the various resonant surfaces in the plasma, whereas p indexes the various velocity harmonics.

2 Critical Island Widths

The critical full island width (in r) which must be exceeded before the electron temperature is flattened within the magnetic separatrix of the magnetic island chain at the kth resonant surface is

$$\frac{W_{T_e\,k}}{r_k} = 5.07\left(\frac{\chi_e}{\chi_{\parallel\,e}}\right)_{r_k}^{1/4}\left(\frac{1}{\epsilon\,s\,n}\right)_{r_k}^{1/2}, \tag{36}$$

where $\chi_e(r)$ is the perpendicular electron energy diffusivity profile, $\epsilon = r/R_0$,

$$\chi_{\parallel\,e} = \frac{\chi_{\parallel\,e}^{\rm brag}\,\chi_{\parallel\,e}^{\rm max}}{\chi_{\parallel\,e}^{\rm brag}+\chi_{\parallel\,e}^{\rm max}}, \tag{37}$$

$$\chi_{\parallel\,e}^{\rm brag} = \frac{1.55\,\tau_{ee}\,v_{T\,e}^2}{1+0.38\,Z_{\rm eff}}, \tag{38}$$

$$\chi_{\parallel\,e}^{\rm max} = \frac{2\,R_0\,v_{T\,e}}{n\,s}\,\frac{r_k}{W_{T_e\,k}}. \tag{39}$$

Equation (36) must be solved iteratively for $W_{T_e\,k}/r_k$.

The critical full island width (in r) which must be exceeded before the ion temperature is flattened within the magnetic separatrix of the magnetic island chain at the kth resonant surface is

$$\frac{W_{T_i\,k}}{r_k} = 5.07\left(\frac{\chi_i}{\chi_{\parallel\,i}}\right)_{r_k}^{1/4}\left(\frac{1}{\epsilon\,s\,n}\right)_{r_k}^{1/2}, \tag{40}$$

where $\chi_i(r)$ is the perpendicular ion energy diffusivity profile,

$$\chi_{\parallel\,i} = \frac{\chi_{\parallel\,i}^{\rm brag}\,\chi_{\parallel\,i}^{\rm max}}{\chi_{\parallel\,i}^{\rm brag}+\chi_{\parallel\,i}^{\rm max}}, \tag{41}$$

$$\chi_{\parallel\,i}^{\rm brag} = 1.95\,\tau_{ii}\,v_{T\,i}^2, \tag{42}$$

$$\chi_{\parallel\,i}^{\rm max} = \frac{2\,R_0\,v_{T\,i}}{n\,s}\,\frac{r_k}{W_{T_i\,k}}. \tag{43}$$

Equation (40) must be solved iteratively for $W_{T_i\,k}/r_k$.

The critical full island width (in r) which must be exceeded before the electron density is flattened within the magnetic separatrix of the magnetic island chain at the kth resonant surface is

$$\frac{W_{n_e\,k}}{r_k} = 5.07\left(\frac{D_\perp}{\chi_{\parallel\,i}}\right)_{r_k}^{1/4}\left(\frac{1}{\epsilon\,s\,n}\right)_{r_k}^{1/2}, \tag{44}$$

where $D_\perp(r)$ is the perpendicular particle diffusivity profile,

$$\chi_{\parallel\,i} = \frac{\chi_{\parallel\,i}^{\rm brag}\,\chi_{\parallel\,i}^{\rm max}}{\chi_{\parallel\,i}^{\rm brag}+\chi_{\parallel\,i}^{\rm max}}, \tag{45}$$

$$\chi_{\parallel\,i}^{\rm brag} = 1.95\,\tau_{ii}\,v_{T\,i}^2, \tag{46}$$

$$\chi_{\parallel\,i}^{\rm max} = \frac{2\,R_0\,v_{T\,i}}{n\,s}\,\frac{r_k}{W_{n_e\,k}}. \tag{47}$$

Equation (44) must be solved iteratively for $W_{n_e\,k}/r_k$.

3 Resonant Plasma Response Model

Fundamental equation:

$$\left(\hat{W}_k+\hat{\delta}_k\right)\mathcal{S}_k\left(\frac{d}{d\hat{t}}+{\rm i}\,\hat{\varpi}_k\right)\Psi_k = f_k\,\Psi_k+\sum_{k'=1,K}E_{kk'}\,\Psi_{k'}+\varDelta_{kw}\,\Psi_{w\,k}+\hat{E}_{kk}\,\chi_k, \tag{48}$$

$$\hat{\tau}_w\,\frac{d\Psi_{w\,k}}{d\hat{t}} = \varDelta_{kw}\,\Psi_{w\,k}+\varSigma_{kw}\,\Psi_k, \tag{49}$$

where $\hat{\tau}_w = \tau_w/\tau_A$,

$$\mathcal{S}_k = \frac{\tau_R(\hat{r}_k)}{\tau_A}, \tag{50}$$

$$\hat{W}_k = \frac{2\,\mathcal{I}}{\hat{a}\,\hat{r}_k}\left(\frac{q}{g\,s}\right)_{\hat{r}_k}^{1/2}|\hat{\Psi}_k|^{\,1/2}, \tag{51}$$

$$\hat{\delta}_k = \frac{\delta_{\rm linear}(\hat{r}_k)}{R_0\,\hat{a}\,\hat{r}_k}, \tag{52}$$

$$f_k = f_{b\,k}+f_{c\,k}+f_{p\,k}+f_{w\,k}+f_{s\,k}, \tag{53}$$

with $\mathcal{I}=0.8227$, and

$$\tau_R(r) = \mu_0\,r^{\,2}\,Q_{00}^{ee}\,\sigma_{ee}, \tag{54}$$

$$\sigma_{ee}(r) = \frac{n_e\,e^{\,2}\,\tau_{ee}}{m_e}. \tag{55}$$

Furthermore,

$$f_{b\,k} = f_{b\,e\,k}+f_{b\,i\,k}, \tag{56}$$

$$f_{b\,e\,k} = \alpha_{b\,e\,k}\left(f_{b\,T_e\,k}+f_{b\,n_e\,k}\right), \tag{57}$$

$$f_{b\,i\,k} = \alpha_{b\,i\,k}\left(f_{b\,T_i\,k}+f_{b\,n_i\,k}\right), \tag{58}$$

$$f_{b\,T_e\,k} = \left(\frac{\eta_e}{1+\eta_e}\right)_{r_k}\frac{\hat{W}_k}{\hat{W}_{T_e\,k}^2+\hat{\rho}_{\theta\,e\,k}^2+\hat{W}_k^{\,2}}, \tag{59}$$

$$f_{b\,n_e\,k} = \left(\frac{1}{1+\eta_e}\right)_{r_k}\frac{\hat{W}_k}{\hat{W}_{n_e\,k}^{\,2}+\hat{\rho}_{\theta\,e\,k}^2+\hat{W}_k^{\,2}}, \tag{60}$$

$$f_{b\,T_i\,k} = \left(\frac{\eta_i}{1+\eta_i}\right)_{r_k} \frac{\hat{W}_k}{\hat{W}_{T_i\,k}^2 + \hat{\rho}_{\theta\,i\,k}^2 + \hat{W}_k^2}, \tag{61}$$

$$f_{b\,n_i\,k} = \left(\frac{1}{1+\eta_i}\right)_{r_k} \frac{\hat{W}_k}{\hat{W}_{n_e\,k}^2 + \hat{\rho}_{\theta\,i\,k}^2 + \hat{W}_k^2}, \tag{62}$$

$$f_{c\,k} = \alpha_{c\,k}\left(f_{c\,T_e\,k} + f_{c\,n_e\,k} + f_{c\,T_i\,k} + f_{c\,n_i\,k}\right), \tag{63}$$

$$f_{c\,T_e\,k} = \left(\frac{n_e}{n_e+n_i}\,\frac{\eta_e}{1+\eta_e}\right)_{r_k} \frac{\hat{W}_k}{\hat{W}_{T_e\,k}^2 + \hat{W}_k^2}, \tag{64}$$

$$f_{c\,n_e\,k} = \left(\frac{n_e}{n_e+n_i}\,\frac{1}{1+\eta_e}\right)_{r_k} \frac{\hat{W}_k}{\hat{W}_{n_e\,k}^2 + \hat{W}_k^2}, \tag{65}$$

$$f_{c\,T_i\,k} = \left(\frac{n_i}{n_e+n_i}\,\frac{\eta_i}{1+\eta_i}\right)_{r_k} \frac{\hat{W}_k}{\hat{W}_{T_i\,k}^2 + \hat{W}_k^2}, \tag{66}$$

$$f_{c\,n_i\,k} = \left(\frac{n_i}{n_e+n_i}\,\frac{1}{1+\eta_i}\right)_{r_k} \frac{\hat{W}_k}{\hat{W}_{n_e\,k}^2 + \hat{W}_k^2}, \tag{67}$$

$$f_{p\,k} = \alpha_{p\,k}\left(f_{p\,T_i\,k} + f_{p\,n_i\,k}\right), \tag{68}$$

$$f_{p\,T_i\,k} = \left(\frac{\eta_i}{1+\eta_i}\right)_{r_k} \frac{\hat{W}_k}{(\hat{W}_{T_i\,k}^2 + \hat{W}_k^2)^2}, \tag{69}$$

$$f_{p\,n_i\,k} = \left(\frac{1}{1+\eta_i}\right)_{r_k} \frac{\hat{W}_k}{(\hat{W}_{n_e\,k}^2 + \hat{W}_k^2)^2}, \tag{70}$$

$$\hat{W}_{T_e\,k} = \frac{\mathcal{I}\,W_{T_e\,k}}{2\,r_k}, \tag{71}$$

$$\hat{W}_{T_i\,k} = \frac{\mathcal{I}\,W_{T_i\,k}}{2\,r_k}, \tag{72}$$

$$\hat{W}_{n_e\,k} = \frac{\mathcal{I}\,W_{n_e\,k}}{2\,r_k}, \tag{73}$$

$$\hat{\rho}_{\theta\,e\,k} = \left(\frac{\mathcal{I}\,\rho_{\theta\,e}}{2\,r}\right)_{r_k} = \left(\frac{2\,\mathcal{I}\,v_{T\,e}\,m_e\,q\,R_0}{e\,B_0\,g\,r^2}\right)_{r_k}, \tag{74}$$

$$\hat{\rho}_{\theta\,i\,k} = \left(\frac{\mathcal{I}\,\rho_{\theta\,i}}{2\,r}\right)_{r_k} = \left(\frac{2\,\mathcal{I}\,v_{T\,i}\,m_i\,q\,R_0}{e\,B_0\,g\,r^2}\right)_{r_k}, \tag{75}$$

$$\alpha_{b\,e\,k} = -2\,\mathcal{I}\,I_g\left(\frac{\omega_{*\,e} + \omega_{\mathrm{nc}\,e}}{\omega_\beta}\right)_{r_k}, \tag{76}$$

$$\alpha_{b\,i\,k} = 2\,\mathcal{I}\,I_g\left[\frac{(n_i/n_e)\,(\omega_{*\,i} + \omega_{\mathrm{nc}\,i}) + (Z_I\,n_I/n_e)\,(\omega_{*\,I} + \omega_{\mathrm{nc}\,I})}{\omega_\beta}\right]_{r_k}, \tag{77}$$

$$\alpha_{c\,k} = 2\,\mathcal{I}\,I_g\,D_R(r_k), \tag{78}$$

$$\alpha_{p\,k} = 8\,\mathcal{I}^3\,I_p\left[\frac{(\omega_{\ast\,i}+\omega_{\mathrm{nc}\,i})\,\omega_{\mathrm{nc}\,i}}{\omega_\beta\,\omega_\Omega}\right]_{r_k}, \tag{79}$$

$$\omega_\beta(r) = \frac{s\,g\,B_0}{\mu_0\,n_e\,e\,R_0^2\,q}, \tag{80}$$

$$\omega_\Omega(r) = \frac{e\,g\,B_0\,s\,q}{m_i}, \tag{81}$$

$$f_{w\,k} = \frac{\Sigma_{kw}^2}{\Delta_{kw}}, \tag{82}$$

$$f_{s\,k} = -\left(\frac{f_{s\,1\,k}\,\hat{W}_{T_e\,k}^2 + f_{s\,2\,k}\,\hat{W}_k^2}{\hat{W}_{T_e\,k}^2+\hat{W}_k^2}\right), \tag{83}$$

$$f_{s\,1\,k} = P_{1\,k}\,\frac{W_k}{r_k}\,\ln\left(\frac{r_k}{W_k}\right) + P_{2\,k}\,\frac{W_k}{r_k}, \tag{84}$$

$$f_{s\,2\,k} = P_{3\,k}\,\frac{W_k}{r_k}, \tag{85}$$

$$P_{1\,k} = 0.41\,A_k^2, \tag{86}$$

$$P_{2\,k} = 0.41\left(A_k^2\left[\ln\left(\frac{1}{r_k}\right)+4.85\right] - \frac{\Sigma_k^{nw}\,A_k}{2} - B_k - 0.44\,A_k\right), \tag{87}$$

$$P_{3\,k} = 0.8\,A_k^2 - 0.27\,B_k - 0.09\,A_k, \tag{88}$$

$$A_k = -\left(\frac{r\,q\,\hat{J}_\phi'}{s}\right)_{r_k}, \tag{89}$$

$$B_k = -\left(\frac{r^{\,2}\,q\,\hat{J}_\phi''}{s}\right)_{r_k}. \tag{90}$$

Here, $I_g = 1.58$ and $I_p = 1.38$.

Let $\Psi_k = R_0\,B_0\,\hat{\Psi}_k\,\mathrm{e}^{-\mathrm{i}\,\varphi_k}$, $\Psi_{w\,k} = R_0\,B_0\,\hat{\Psi}_{w\,k}\,\mathrm{e}^{-\mathrm{i}\,\varphi_{w\,k}}$, $\chi_k = R_0\,B_0\,\hat{\chi}_k\,\mathrm{e}^{-\mathrm{i}\,\zeta_k}$, and $E_{kk'} = \hat{E}_{kk'}\,\mathrm{e}^{-\mathrm{i}\,\xi_{kk'}}$. Let

$$X_k = \hat{\Psi}_k\,\cos\varphi_k, \tag{91}$$

$$Y_k = \hat{\Psi}_k\,\sin\varphi_k, \tag{92}$$

$$U_k = \hat{\Psi}_{w\,k}\,\cos\varphi_{w\,k}, \tag{93}$$

$$V_k = \hat{\Psi}_{w\,k}\,\sin\varphi_{w\,k}. \tag{94}$$

So, Eqs. (48) and (49) give

$$\left(\hat{W}_k+\hat{\delta}_k\right)\mathcal{S}_k\left(\frac{d}{d\hat{t}}+\mathrm{i}\,\hat{\varpi}_k\right)(X_k-\mathrm{i}\,Y_k) = f_k\,(X_k-\mathrm{i}\,Y_k) \tag{95}$$

$$+ \sum_{k'=1,K} \hat{E}_{kk'}\,(\cos\xi_{kk'} - \mathrm{i}\,\sin\xi_{kk'})\,(X_{k'} - \mathrm{i}\,Y_{k'})$$

$$+ \Sigma_{kw}\,(U_k - \mathrm{i}\,V_k) + \hat{E}_{kk}\,\hat{\chi}_k\,(\cos\zeta_k - \mathrm{i}\,\sin\zeta_k),$$

$$\hat{\tau}_w\,\frac{d}{d\hat{t}}\,(U_k - \mathrm{i}\,V_k) = \Delta_{kw}\,(U_k - \mathrm{i}\,V_k) + \Sigma_{kw}\,(X_k - \mathrm{i}\,Y_k).$$

Follows that

$$\left(\hat{W}_k + \hat{\delta}_k\right)\mathcal{S}_k\left(\frac{dX_k}{d\hat{t}} + \hat{\varpi}_k\,Y_k\right) = f_k\,X_k + \sum_{k'=1,K} \hat{E}_{kk'}\,(\cos\xi_{kk'}\,X_{k'} - \sin\xi_{kk'}\,Y_{k'}) \tag{96}$$

$$+ \Sigma_{kw}\,U_k + \hat{E}_{kk}\,\hat{\chi}_k\,\cos\zeta_k,$$

$$\left(\hat{W}_k + \hat{\delta}_k\right)\mathcal{S}_k\left(\frac{dY_k}{d\hat{t}} - \hat{\varpi}_k\,X_k\right) = f_k\,Y_k + \sum_{k'=1,K} \hat{E}_{kk'}\,(\cos\xi_{kk'}\,Y_{k'} + \sin\xi_{kk'}\,X_{k'}) \tag{97}$$

$$+ \Sigma_{kw}\,V_k + \hat{E}_{kk}\,\hat{\chi}_k\,\sin\zeta_k,$$

$$\hat{\tau}_w\,\frac{dU_k}{d\hat{t}} = \Delta_{kw}\,U_k + \Sigma_{kw}\,X_k, \tag{98}$$

$$\hat{\tau}_w\,\frac{dV_k}{d\hat{t}} = \Delta_{kw}\,V_k + \Sigma_{kw}\,Y_k. \tag{99}$$

where

$$\hat{W}_k = \frac{2\,\mathcal{I}}{\hat{a}\,\hat{r}_k}\left(\frac{q}{g\,s}\right)^{1/2}_{\hat{r}_k}(X_k^{\,2} + Y_k^{\,2})^{\,1/4}. \tag{100}$$

Note, finally, that

$$\delta\hat{T}_k = \sum_{k'\neq k} \hat{E}_{kk'}\,\hat{\Psi}_k\,\hat{\Psi}_{k'}\,\sin(\varphi_k - \varphi_{k'} - \xi_{kk'}) + \Sigma_{kw}\,\hat{\Psi}_k\,\hat{\Psi}_{w\,k}\,\sin(\varphi_k - \varphi_{w\,k})$$

$$+ \hat{E}_{kk}\,\hat{\Psi}_k\,\hat{\chi}_k\,\sin(\varphi_k - \zeta_k), \tag{101}$$

or

$$\delta\hat{T}_k = \sum_{k'=1,K} \hat{E}_{kk'}\,[(Y_k\,X_{k'} - X_k\,Y_{k'})\,\cos\xi_{kk'} - (X_k\,X_{k'} + Y_k\,Y_{k'})\,\sin\xi_{kk'}]$$

$$+ \Sigma_{kw}\,(Y_k\,U_k - X_k\,V_k) + \hat{E}_{kk}\,\hat{\chi}_k\,(Y_k\,\cos\zeta_k - X_k\,\sin\zeta_k). \tag{102}$$

4 Coil Optimization

Assuming an ideal response of the plasma to the applied RMP, characterized by $\Psi_k = 0$ for all k, we have

$$\Delta\Psi_k = |E_{kk}|\left(I_U\,\mathrm{e}^{-\mathrm{i}\,\Delta_U}\,\chi_k^U + I_M\,\chi_k^M + I_L\,\mathrm{e}^{\,\mathrm{i}\,\Delta_L}\,\chi_k^L\right), \tag{103}$$

where U, M, and L correspond to the upper, middle, and lower coil-sets, respectively, $I_{U,M,L}$ and $\Delta_{U,M,L}$ are the (real) amplitudes and helical phase-shifts of the currents flowing in the upper, middle, and lower coil-sets, respectively, with $\Delta_M = 0$. Finally, $\chi_k^{U,M,L}$ = are the (complex) χ_k values when 1 kA flows in the upper, middle, and lower coil-sets, respectively, with the same helical phase as the current flowing in the middle coil set. Let

$$z_k = \frac{\Delta\Psi_k}{|E_{kk}|\,(I_U + I_M + I_L)}, \tag{104}$$

$$\lambda_U = \frac{I_U}{I_U + I_M + I_L}, \tag{105}$$

$$\lambda_L = \frac{I_L}{I_U + I_M + I_L}, \tag{106}$$

where $I_U \geq 0$, $I_M \geq 0$, and $I_U + I + M \leq 1$. It follows that

$$z_k = \lambda_U\,\chi_k^U\,\mathrm{e}^{-\mathrm{i}\,\Delta_U} + (1-\lambda_U-\lambda_L)\,\chi_k^M + \lambda_L\,\chi_k^L\,\mathrm{e}^{\mathrm{i}\,\Delta_L}. \tag{107}$$

Let

$$\chi_k^U\,\chi_k^{L\,*} = |\chi_k^U\,|\chi_k^L|\,\mathrm{e}^{\mathrm{i}\,\gamma_k^{UL}}, \tag{108}$$

$$\chi_k^U\,\chi_k^{M\,*} = |\chi_k^U|\,|\chi_k^M|\,\mathrm{e}^{\mathrm{i}\,\gamma_k^{UM}}, \tag{109}$$

$$\chi_k^M\,\chi_k^{L\,*} = |\chi_k^M|\,|\chi_k^L|\,\mathrm{e}^{\mathrm{i}\,\gamma_k^{ML}}. \tag{110}$$

Follows that

$$\begin{aligned} |z_k|^2 = {}& \lambda_U^2\,|\chi_k^U|^2 + (1-\lambda_U-\lambda_L)^2\,|\chi_k^M|^2 + \lambda_L^2\,|\chi_k^L|^2 \\ &+ 2\,\lambda_U\,(1-\lambda_U-\lambda_L)\,|\chi_k^U|\,|\chi_k^M|\,\cos(\gamma_k^{UM} - \Delta^U) && (111)\\ &+ 2\,(1-\lambda_U-\lambda_L)\,\lambda_L\,|\chi_k^M|\,|\chi_k^L|\,\cos(\gamma_k^{ML} - \Delta^L) && (112)\\ &+ 2\,\lambda_U\,\lambda_L\,|\chi_k^U|\,|\chi_k^L|\,\cos(\gamma_k^{UL} - \Delta^U - \Delta^L). && (113) \end{aligned}$$

Suppose that $k = 1$ corresponds to the innermost resonant surface. Let the k'th and $(k'+1)$th resonant surfaces straddle the top of the pedestal. Let

$$f_1 = |z_1|^2, \tag{114}$$

$$f_2 = \frac{(\Psi_{N\,k'+1} - \Psi_N)\,|z_k'|^2 + (\Psi_N - \Psi_{N\,k'})\,|z_{k'+1}|^2}{\Psi_{N\,k'+1} - \Psi_{N\,k'}}. \tag{115}$$

Wish to simultaneously minimize f_1 and maximize f_2. Minimize

$$f(\lambda_U, \lambda_L, \Delta_U, \Delta_L) = w_1\,f_1 + \frac{1-w_1}{f_2}, \tag{116}$$

where $0 \leq w_1 \leq 1$.