Magnetic Reconnection in Tokamaks

Richard Fitzpatrick

Institute for Fusion Studies
University of Texas at Austin
Austin TX, USA

Lectures available online at
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Outline

1. Introduction.
2. Toroidal magnetic confinement.
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4. Profile modification.
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7. Subsonic island theory.
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1: Introduction
Magnetic Reconnection in Astrophysical Plasmas

- Narrow current sheets. Highly unstable to tearing instabilities.
- Very rapid (quasi-Alfvénic) reconnection rates.
- Reconnection gives rise to significant release of thermal energy, as well as copious charged particle acceleration.
- Main aims of astrophysical reconnection theory are to explain rapid onset of reconnection, to account for fast reconnection rate, and to understand energy release and particle acceleration mechanisms.
Magnetic Reconnection\(^a\) in Tokamak Plasmas

- Extended current distributions. Comparatively stable to tearing instabilities.
- Reconnection changes topology of magnetic flux-surfaces, thereby degrading energy and particle confinement.
- Reconnection-induced energy release and charged particle acceleration completely negligible.
- Reconnection timescale irrelevant since sufficient time for tearing instability growing at smallest conceivable rate to eventually cause significant change in topology of flux-surfaces.
- Main aims of fusion reconnection theory are to understand various factors that determine stability, and final saturated amplitude, of tearing instabilities.

\(^a\)Excluding sawtooth oscillation.
2: Toroidal Magnetic Confinement
Principles of Tokamak Confinement

- Tokamaks designed to trap hot plasma on set of axisymmetric, nested, toroidal magnetic flux-surfaces.\(^a\)
- Fundamental principle—charged particles free to stream along magnetic field-lines, but “stick” to flux-surfaces due to their (relatively) small gyroradii.
- Heat/particles flow rapidly along field-lines, but can only diffuse relatively slowly across flux-surfaces. Diffusion rate controlled by small-scale plasma turbulence.

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Poloidal Cross-Section of Toroidal Confinement Device

Rapid flow of heat/particles along field-lines

Magnetic flux-surface

Particle gyroradius

Slow diffusion of heat/particles across flux-surfaces
Macroscopic Instabilities

- Two main types of macroscopic instability\(^a\) in tokamaks:
  - Catastrophic “ideal” (\textit{i.e.}, non-reconnecting) instabilities that disrupt plasma in matter of micro-seconds—easily avoided.
  - Slowly growing “tearing” instabilities that reconnect magnetic flux-surfaces, but eventually saturate at relatively low amplitude to form \textit{magnetic islands}, thereby degrading plasma confinement—much harder to avoid.

\(^a\textit{MHD Instabilities}, \text{G. Bateman (MIT, 1978).}\)
Magnetic Islands

- Centered on *rational flux-surfaces* which satisfy $\vec{k} \cdot \vec{B} = 0$, where $\vec{k}$ is wave-number of mode, and $\vec{B}$ is equilibrium magnetic field.
- Effectively “short-circuit” confinement by allowing heat/particles to transit island region by rapidly flowing along field-lines, rather than slowly diffusing across flux-surfaces.
Need for Magnetic Island Theory

• Magnetic island formation associated with *nonlinear* phase of tearing mode growth (*i.e.*, when island width becomes greater than linear layer width at rational surface).

• In very hot plasmas found in modern-day tokamaks, linear layers so thin that tearing mode already in nonlinear regime when first detected.

• Linear tearing mode theory largely irrelevant. Require nonlinear magnetic island theory to explain experimental observations.
3: MHD Theory
Introduction

• Tearing modes are macroscopic instabilities that affect whole plasma. Natural to investigate them using some form of fluid-theory.

• Simplest fluid theory is well-known magnetohydrodynamical approximation, which effectively treats plasma as single-fluid.

• Use slab approximation to simplify analysis.

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\(^a\) Plasma Confinement, R.D. Hazeltine, and J.D. Meiss (Dover, 2003).
Slab Approximation

"toroidal" $z$

"poloidal" $y$

$x = 0$

d/dz = 0

perfectly conducting wall

rational surface

periodic in y–dirn.
**Slab Model**

- Cartesian coordinates: \((x, y, z)\). Let \(\partial/\partial z \equiv 0\).
- Assume presence of dominant uniform “guide-field” \(\vec{B}_z \vec{e}_z\).
- All field-strengths normalized to \(B_z\).
- All lengths normalized to equilibrium magnetic shear-length:
  
  \[ L_s = B_z / (dB_y^{(0)}/dx)_{x=0}. \]

- All times normalized to Alfvén time calculated with \(B_z\).
- Perfect wall boundary conditions at \(x = \pm a\).
- Wave-number of tearing instability: \(\vec{k} = (0, k, 0)\), so \(\vec{k} \cdot \vec{B} = 0\) at \(x = 0\). Hence, rational surface at \(x = 0\).
Model MHD Equations

• Let $\vec{B} = \nabla \psi \times \vec{e}_z + B_z \vec{e}_z$, $\vec{V} = \nabla \phi \times \vec{e}_z$, where $\vec{V}$ is $E \times B$ vely.

• $\vec{B} \cdot \nabla \psi = \vec{V} \cdot \nabla \phi = 0$, so $\psi$ maps magnetic flux-surfaces, and $\phi$ maps stream-lines of $E \times B$ fluid.

• Incompressible MHD equations: \(^a\)

$$\frac{\partial \psi}{\partial t} = [\phi, \psi] + \eta J,$$

$$\frac{\partial U}{\partial t} = [\phi, U] + [J, \psi] + \mu \nabla^2 U,$$

where $J = \nabla^2 \psi$, $U = \nabla^2 \phi$, and $[A, B] = A_x B_y - A_y B_x$. Here, $\eta$ is resistivity, and $\mu$ is viscosity. In normalized units: $\eta, \mu \ll 1$.

• First equation is $z$-component of Ohm’s law. Second equation is $z$-component of curl of plasma equation of motion.

\(^a\) Plasma Confinement, R.D. Hazeltine, and J.D. Meiss (Dover, 2003).
Outer Region

- In “outer region”, which comprises most of plasma, can neglect non-linear, non-ideal ($\eta$ and $\mu$), and inertial ($\partial/\partial t$ and $\vec{V} \cdot \nabla$) effects.

- Vorticity equation reduces to

  \[ [J, \psi] \simeq 0. \]

- When linearized, obtain $\psi(x, y) = \psi^{(0)}(x) + \psi^{(1)}(x) \cos(ky)$, where $B_y^{(0)} = -d\psi^{(0)}/dx$, and

  \[ \left( \frac{d^2}{dx^2} - k^2 \right) \psi^{(1)} - \left( \frac{d^2 B_y^{(0)}}{B_y^{(0)}} \right) \psi^{(1)} = 0. \]

- Equation is singular at rational surface, $x = 0$, where $B_y^{(0)} = 0$. 
A rational surface is shown with critical points at $x = -a$, $x = 0$, and $x = +a$. The vertical line $x = 0$ is indicated as a tearing eigenfunction. The region $W$ is highlighted above the surface.
Tearing Stability Index

- Find tearing eigenfunction, \( \psi^{(1)}(x) \), which is continuous, has tearing parity \([\psi^{(1)}(-x) = \psi^{(1)}(x)]\), and satisfies boundary condition \( \psi^{(1)}(a) = 0 \) at conducting wall.

- In general, eigenfunction has gradient discontinuity across rational surface (at \( x = 0 \)). Allowed because tearing mode equation singular at rational surface.

- Tearing stability index:

\[
\Delta' = \left[ \frac{d \ln \psi^{(1)}}{dx} \right]_{0-}^{0+}.
\]

- According to conventional MHD theory, a tearing mode is unstable if \( \Delta' > 0 \).

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Inner Region

- “Inner region” centered on rational surface, \( x = 0 \). Of extent, \( W \ll 1 \), where \( W \) is magnetic island width (in \( x \)).

- In inner region, non-ideal effects, non-linear effects, and plasma inertia can all be important.

- Inner solution must be asymptotically matched to outer solution already obtained.
**Constant-ψ Approximation**

- $\psi^{(1)}(x)$ generally does not vary significantly in $x$ over inner region:

  \[ |\psi^{(1)}(W) - \psi^{(1)}(0)| \ll |\psi^{1}(0)|. \]

- *Constant-ψ approximation*: treat $\psi^{(1)}(x)$ as constant in $x$ over inner region.

- Approximation valid provided

  \[ |\Delta'| W \ll 1, \]

  which is easily satisfied for conventional tearing modes in tokamak plasmas.
Constant-$\psi$ Magnetic Island

- In vicinity of rational surface, $\psi^{(0)} \to -x^2/2$, so
  $$\psi(x, y, t) \simeq -x^2/2 + \Psi(t) \cos \theta,$$
  where $\Psi = \psi^{(1)}(0)$ is “reconnected flux”, and $\theta = ky$.

- Full island width, $W = 4\sqrt{\Psi}$.

![Diagram showing $x$-point and $O$-point with $\Psi = +\Psi$ and $\psi = -\Psi$ on separatrices.](image-url)
Flux-Surface Average Operator

- Flux-surface average operator is annihilator of Poisson bracket 
  \([A, \psi] \equiv \vec{B} \cdot \nabla A \equiv k_x (\partial A / \partial \theta)_\psi\) for any \(A\): i.e.,

  \[\langle [A, \psi] \rangle \equiv 0.\]

- Outside separatrix:

  \[
  \langle f(\psi, \theta) \rangle = \oint f(\psi, \theta) \frac{d\theta}{2\pi}.
  \]

- Inside separatrix:

  \[
  \langle f(s, \psi, \theta) \rangle = \int_{-\theta_0}^{\theta_0} \frac{f(s, \psi, \theta) + f(-s, \psi, \theta)}{2 |x|} \frac{d\theta}{2\pi},
  \]

  where \(s = \text{sgn}(x)\), and \(x(s, \psi, \theta_0) = 0\).
MHD Flow - 1

• Move to island frame. Look for steady-state solution: $\partial/\partial t = 0$.\textsuperscript{a}

• Ohm’s law:

\[
0 \simeq [\phi, \psi] + \eta J.
\]

• Since $\eta \ll 1$, first term potentially much larger than second.

• To lowest order:

\[
[\phi, \psi] \simeq 0.
\]

• Follows that

\[
\phi = \phi(\psi) :
\]

i.e., MHD flow constrained to be around flux-surfaces.

MHD Flow - II

• Let

\[ M(\psi) = \frac{d\phi}{d\psi}. \]

• Easily shown that

\[ V_y = xM. \]

• By symmetry, \[ M(\psi) \] is odd function of \( x \). Hence,

\[ M = 0 \]

inside separatrix: i.e., no flow inside separatrix in island frame. Plasma \textit{trapped} within magnetic separatrix.
MHD Flow - III

• Vorticity equation:

\[ 0 \simeq [-MU + J, \psi] + \mu \nabla^4 \phi. \]

• Flux-surface average, recalling that \(\langle [A, \psi] \rangle = 0\):

\[ \langle \nabla^4 \phi \rangle \equiv -\frac{d^2}{d\psi^2} \left( \langle x^4 \rangle \frac{dM}{d\psi} \right) \simeq 0. \]

• Solution outside separatrix:

\[ M(\psi) = \text{sgn}(x) M_0 \int_{-\psi}^{\psi} d\psi / \langle x^4 \rangle \left/ \int_{-\psi}^{-\infty} d\psi / \langle x^4 \rangle \right. . \]
MHD Flow - IV

• Note

\[ V_y = x M \rightarrow |x| M_0 \]

as \( |x|/W \rightarrow \infty \).

• V-shaped velocity profile that extends over whole plasma.

• Expect *isolated* magnetic island to have *localized* velocity profile. Suggests that \( M_0 = 0 \) for isolated island.

• Hence, zero MHD flow in island frame: i.e., island propagates at local \( \vec{E} \times \vec{B} \) velocity.
\[ V = -V \]

\[ x = -\alpha \quad x = 0 \quad x = +\alpha \]

Unlocalized profile

Localized profile

Rational surface

\[ V \text{ and } E \times B \]
Rutherford Equation - I

- Asymptotic matching between inner and outer regions yields:

\[ \Delta' \Psi = -4 \int_{-\infty}^{+\psi} \langle J \cos \theta \rangle \, d\psi. \]

- In island frame, in absence of MHD flow, vorticity equation reduces to

\[ [J, \psi] \simeq 0. \]

- Hence,

\[ J = J(\psi). \]
Rutherford Equation - II

• Ohm’s law:

\[ \frac{d\Psi}{dt} \cos \theta \simeq [\phi, \psi] + \eta J(\psi). \]

• Have shown there is no MHD-flow \( i.e., \phi \sim O(1) \), but can still be resistive flow \( i.e., \phi \sim O(\eta) \).

• Eliminate resistive flow by flux-surface averaging:

\[ \frac{d\Psi}{dt} \langle \cos \theta \rangle \simeq \eta J(\psi) \langle 1 \rangle. \]

• Hence,

\[ \Delta' \Psi \simeq -\frac{4}{\eta} \frac{d\Psi}{dt} \int_{-\infty}^{\infty} \frac{\langle \cos \theta \rangle^2}{\langle 1 \rangle} d\psi. \]
Rutherford Equation - III

• Use $W = 4 \sqrt{\Psi}$, and evaluate integral. Obtain Rutherford island width evolution equation:\textsuperscript{a}

$$\frac{0.823}{\eta} \frac{dW}{dt} \approx \Delta'.$$

• According to Rutherford equation, island grows algebraically on resistive time-scale.

• Rutherford equation does not predict island saturation.

\textsuperscript{a}P.H. Rutherford, Phys. Fluids 16, 1903 (1973).
Rutherford Equation - IV

• Higher order asymptotic matching between inner and outer regions yields:

\[
\frac{0.823}{\eta} \frac{dW}{dt} \simeq \Delta' - 0.41 \left( -\frac{d^4 B_y^{(0)}}{dx^4} \right)_{x=0} - 0.41 \left( -\frac{d^2 B_y^{(0)}}{dx^2} \right)_{x=0} W.
\]

• Hence, saturated \((d/dt = 0)\) island width is

\[
W_0 = \frac{\Delta'}{0.41} \left( -\frac{d^2 B_y^{(0)}}{dx^2} \right)_{x=0}.
\]

Main Predictions of MHD Theory

• Tearing mode unstable if $\Delta' > 0$.

• Island propagates at local $\vec{E} \times \vec{B}$ velocity at rational surface.

• Island grows algebraically on resistive time-scale.

• Saturated island width:

$$W_0 = \frac{\Delta'}{0.41} \left( -\frac{d^2 B_y^{(0)}}{dx^2} \right) \left( \frac{d^4 B_y^{(0)}}{dx^4} \right)_{x=0}.$$
4: Profile Modification
Temperature Flattening - I

- For sake of simplicity, assume cold ions. Let $T$ be electron temperature profile (minus uniform background value, $T_e$).
- In immediate vicinity of island, unperturbed profile is

$$T(\chi) \simeq T_e \frac{\chi}{L_T},$$

where $L_T$ is equilibrium temperature gradient scale-length.
- Perturbed temperature profile determined by competition between parallel and perpendicular heat transport:

$$\chi_{||} \nabla_{||}^2 T + \chi_{\perp} \nabla_{\perp}^2 T \simeq 0.$$

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Temperature Flattening - II

- Parallel transport term attempts to make temperature a flux-surface function. Cannot have odd flux-surface function inside island separatrix. So, if $T = T(\psi)$ then temperature is flattened inside separatrix.

- Perpendicular transport term attempts to relax temperature profile to unperturbed profile. Opposes temperature flattening.

- Temperature is flattened inside separatrix if parallel transport term dominates perpendicular transport term.
Temperature Flattening - III

• Have

\[ \nabla_{\parallel} \simeq \frac{kW}{L_s}. \]

• Also

\[ \nabla_{\perp} \simeq \frac{1}{W}. \]

• Hence, parallel transport dominates perpendicular transport when

\[ W \gg W_c, \]

where

\[ kW_c \sim \left( \frac{X_{\perp}}{X_{\parallel}} \right)^{1/4} (kL_s)^{1/2}. \]
Temperature Flattening - IV

• In “collisionless” fusion plasmas, Braginskii\(^a\) expression for parallel heat flux due to conduction impossibly large: \(i.e.,\)

\[
\chi_{\parallel} \nabla_{\parallel} T \gg n_e v_e T,
\]

where \(n_e\) is uniform background electron number density, and \(v_e\) is thermal velocity.

• In this situation, parallel heat transport becomes *convective* in nature, rather than *diffusive*. Can model this effect by “flux limiting” heat flux: \(i.e.,\) replace \(\chi_{\parallel} \nabla_{\parallel} T\) by \(n_e v_e T\).

• Leads to more realistic expression for critical island width\(^b\)

\[
k W_c \sim \left( \frac{\chi_{\perp}}{n_e v_e L_s} \right)^{1/3} (k L_s)^{2/3}.
\]

\(^a\)S.I. Braginskii, Reviews of Plasma Physics, (Consultants Bureau, 1965)

Temperature Flattening - IV

- Temperature profile only flattened inside separatrix when island width exceeds critical value.
- Critical width much smaller than minor radius in conventional tokamak.
- Temperature flattening implies complete loss of radial energy confinement across island.
Density Flattening - I\(^a\)

- Let \( n \) be electron number density (minus uniform background value, \( n_e \)).

- In immediate vicinity of magnetic island, unperturbed density profile is

\[
n(x) \simeq n_e \frac{x}{L_n},
\]

where \( L_n \) is equilibrium density gradient scale-length.

- Sound waves propagating along magnetic field-lines act to make density a flux-surface function. Cannot have odd flux-surface function inside island separatrix. So, if \( n = n(\psi) \) then density is flattened inside separatrix.

Density Flattening - II

- Expect sound waves to flatten density profile when\(^a\)

\[
(\mathbf{V}_* \cdot \nabla) n \ll c_s \nabla_\parallel n,
\]

where \(c_s = \sqrt{T_e/m_i}\) is (unnormalized) \textit{sound speed}, and

\[
\mathbf{V}_* = c_s \frac{\rho}{L_n} \mathbf{e}_y
\]

is \textit{diamagnetic velocity} due to equilibrium density gradient. Furthermore, \(\rho = c_s/(eB_z/m_i)\) is ion Larmor radius calculated with electron temperature.

Density Flattening - III

• Density profile flattened when island width exceeds critical value

\[ W_c \sim \rho \frac{L_s}{L_n}. \]

• In typical tokamak plasma, critical width for density flattening generally considerably larger than that for temperature flattening, but still much smaller than minor radius.

• If island width exceeds critical value for density flattening then pressure profile completely flattened inside island separatrix. Implies complete loss of radial energy and particle confinement across island.
5: Bootstrap Current Destabilization
Neoclassical Effects

- So-called *neoclassical effects*\(^a\) in tokamak plasmas arise from essential *toroidicity* of such plasmas, combined with *extremely long mean-free-path* of electrons and ions streaming along magnetic field-lines, due to high plasma temperature.

Trapped and Passing Particles

**strong toroidal field**

**weak toroidal field**

- **toroidal axis**
- **magnetic mirroring**
- **trapped orbit**
- **passing orbit**
Bootstrap Current - I

• In toroidal plasma, friction between trapped and passing electrons leads to appearance of non-inductive *bootstrap current* in Ohm’s law: a

\[
\frac{d\psi}{dt} \cos \theta \simeq [\phi, \psi] + \eta [J(\psi) - J_{\text{boot}}],
\]

where

\[
J_{\text{boot}} = -1.46 \sqrt{\epsilon} B_\theta^{-1} \frac{dP}{dx}.
\]

Here, \(\epsilon\) is inverse aspect-ratio, \(1.46 \sqrt{\epsilon}\) is measure of fraction of trapped-particles, \(B_\theta\) is poloidal magnetic field-strength, and \(P\) is plasma pressure.

Bootstrap Current - II

- For sufficiently wide island, pressure profile *flattened* inside separatrix.
- Bootstrap current consequently disappears inside separatrix.
- Absence of bootstrap current inside separatrix, and continued presence outside, leads to *destabilizing* term in Rutherford island equation:\(^{a}\)

\[
\frac{0.823}{\eta} \frac{dW}{dt} \simeq \Delta' + 9.25 \sqrt{\epsilon} \beta \frac{L_s}{L_P} \frac{B_z}{B_\theta} \frac{1}{W},
\]

where \(\beta = \mu_0 n_e T_e / B_z^2\), and \(L_P^{-1} = L_n^{-1} + L_T^{-1}\).

Neoclassical Tearing Modes

- A *neoclassical tearing mode* (NTM) is an *intrinsically stable* $(\Delta' < 0)$ tearing mode destabilized by bootstrap term.

- Bootstrap term in Rutherford equation relatively large, especially at small island widths. Would expect plasma to be filled with NTMs, and confinement to be wrecked. \(^a\)

- This is not observed to be case. Experimental evidence for *threshold island width* above which NTMs grow, but below which they decay. \(^b\)

- Suggests presence of *stabilizing effect* in Rutherford equation that opposes destabilizing bootstrap term.

Incomplete Pressure Flattening - I

- Bootstrap destabilization is caused by flattening of pressure profile inside island separatrix. If there is no flattening then there is no destabilization.

- Pressure flattening only occurs when island width exceeds critical value $W_c$.

- When incomplete pressure flattening incorporated into Rutherford equation find that\(^a\)

$$\frac{0.823}{\eta} \frac{dW}{dt} \simeq \Delta' + 9.25 \sqrt{\epsilon} \beta \frac{L_s}{L_p} \frac{B_z}{B_{\theta}} \frac{W}{W^2 + W_c^2}.$$  

Incomplete Pressure Flattening - II

\[ \frac{0.823}{\eta} \frac{dW}{dt} \]

complete pressure flattening

threshold NTM width

saturated NTM width

\[ \Delta' \]

\[ W \]
6: Drift-MHD Theory
**Introduction**

- In drift-MHD model, which is far more accurate representation of tokamak plasma than MHD model, analysis retains *charged particle drift velocities*, in addition to \( \mathbf{E} \times \mathbf{B} \) velocity.
- Essentially *two-fluid* theory of plasma.
- Characteristic length-scale, \( \rho \), is *ion Larmor radius calculated with electron temperature*.
- Characteristic velocity is *diamagnetic velocity*, \( V_* \).
- Normalize all lengths to \( \rho \), and all velocities to \( V_* \).
Basic Assumptions

- Retain slab model, for sake of simplicity.
- Assume island sufficiently wide that $T = T(\psi)$.
- Assume $T_i/T_e = \tau = \text{constant}$, for sake of simplicity, where $T_i$ and $T_e$ are complete electron and ion temperature profiles.
Basic Definitions

• Variables:
  - $\psi$ - magnetic flux-function.
  - $J$ - parallel current density.
  - $\phi$ - guiding-center (i.e., MHD) stream-function.
  - $U$ - parallel ion vorticity.
  - $n$ - electron number density (minus uniform background).
  - $V_z$ - parallel ion velocity.

• Parameters:
  - $\alpha = (L_n/L_s)^2$, where $L_s$ is magnetic shear length, and $L_n$ is density gradient scale-length.
  - $\eta$ - resistivity. $D$ - (perpendicular) particle diffusivity. $\mu_{i/e}$ - (perpendicular) ion/electron viscosity.
Drift-MHD Equations - I

- Steady-state drift-MHD equations:

\[ \psi = -\chi^2/2 + \Psi \cos \theta, \quad U = \nabla^2 \phi, \]
\[ 0 = [\phi - n, \psi] + \eta J, \]
\[ 0 = [\phi, U] - \frac{\tau}{2} \left\{ \nabla^2 [\phi, n] + [U, n] + [\nabla^2 n, \phi] \right\} \]
\[ + [J, \psi] + \mu_i \nabla^4 (\phi + \tau n) + \mu_e \nabla^4 (\phi - n), \]
\[ 0 = [\phi, n] + [V_z + J, \psi] + D \nabla^2 n, \]
\[ 0 = [\phi, V_z] + \alpha [n, \psi] + \mu_i \nabla^2 V_z. \]

Drift-MHD Equations - II

• Symmetry: \( \psi, J, V_z \) even in \( x \). \( \phi, n, U \) odd in \( x \).

• Boundary conditions as \( |x|/W \rightarrow \infty \):
  - \( n \rightarrow -(1 + \tau)^{-1} x \).
  - \( \phi \rightarrow -V x \).
  - \( J, U, V_z \rightarrow 0 \).

• Here, \( V \) is island phase-velocity in \( \vec{E} \times \vec{B} \) frame.

• \( V = 1 \) corresponds to island propagating with electron fluid.
  \( V = -\tau \) corresponds to island propagating with ion fluid.

• Expect

\[
1 \gg \alpha \gg \eta, D, \mu_i, \mu_e.
\]
Electron Fluid

• Ohm’s law:

\[ 0 = [\phi - n, \psi] + \eta J. \]

• Since \( \eta \ll 1 \), first term potentially much larger than second.

• To lowest order:

\[ [\phi - n, \psi] \approx 0. \]

• Follows that

\[ n = \phi + H(\psi) : \]

\textit{i.e.}, electron stream-function \( \phi_e = \phi - n \) is \textit{flux-surface function}. Electron fluid flow constrained to be around flux-surfaces.
Sound Waves

• Parallel flow equation:

\[ 0 = [\phi, V_z] + \alpha [n, \psi] + \mu_i \nabla^2 V_z. \]

• Highlighted term dominant provided

\[ W \gg \alpha^{-1/2} = L_s/L_n. \]

• If this is case then, to lowest order,

\[ n = n(\psi), \]

which implies \( n = 0 \) inside separatrix.

• So, if island sufficiently wide then sound-waves able to flatten density profile inside island separatrix.
Subsonic vs. Supersonic Islands

- Wide islands satisfying

\[ W \gg L_s/L_n \]

termed subsonic islands. Expect such islands to exhibit flattened density profile within separatrix. Subsonic islands strongly coupled to both electron and ion fluids.

- Narrow islands satisfying

\[ W \ll L_s/L_n \]

termed supersonic islands. No flattening of density profile within separatrix. Supersonic islands strongly coupled to electron fluid, but only weakly coupled to ion fluid.
7: Subsonic Island Theory
Introduction

• To lowest order:
  \[ \phi = \phi(\psi), \quad n = n(\psi). \]
  
• Follows that both electron stream-function, \( \phi_e = \phi - n \), and ion stream-function, \( \phi_i = \phi + \tau n \), are flux-surface functions. Both electron and ion fluid flow constrained to follow flux-surfaces.

• Let
  \[ M(\psi) = \frac{d\phi}{d\psi}, \quad L(\psi) = \frac{dn}{d\psi}. \]

• Follows that
  \[ V_{E \times B, y} = \chi M, \quad V_{e, y} = \chi (M - L), \quad V_{i, y} = \chi (M + \tau L). \]

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Density Flattening

• By symmetry, both $M(\psi)$ and $L(\psi)$ are odd functions of $x$.
  Hence,

  $$M(\psi) = L(\psi) = 0$$

  inside separatrix: i.e., no electron/ion flow within separatrix in island frame.

• Electron/ion fluids constrained to propagate with island inside separatrix.

• Density profile flattened within separatrix.
Analysis - I

- Density equation reduces to

\[ 0 \simeq [V_z + J, \psi] + D \nabla^2 n. \]

- Vorticity equation reduces to

\[ 0 \simeq \left[ -M U - \left( \tau/2 \right)(L U + M \nabla^2 n) + J, \psi \right] \\
+ \mu_i \nabla^4 (\phi + \tau n) + \mu_e \nabla^4 (\phi - n). \]

- Flux-surface average both equations, recalling that \( \langle [A, \psi] \rangle = 0. \)
Analysis - II

• Obtain

\[ \langle \nabla^2 n \rangle \simeq 0, \]

and

\[ (\mu_i + \mu_e) \langle \nabla^4 \phi \rangle + (\mu_i \tau - \mu_e) \langle \nabla^4 n \rangle \simeq 0. \]

• Solution outside separatrix:

\[ M(\psi) = -\frac{(\mu_i \tau - \mu_e)}{(\mu_i + \mu_e)} L(\psi) + F(\psi), \]

where

\[ L(\psi) = -\text{sgn}(x) L_0 / \langle x^2 \rangle, \]

and \( F(\psi) \) is previously obtained MHD profile:

\[ F(\psi) = \text{sgn}(x) F_0 \int_{-\psi}^{\psi} \frac{d\psi}{\langle x^4 \rangle} \bigg/ \int_{-\psi}^{-\infty} \frac{d\psi}{\langle x^4 \rangle}. \]
Velocity Profiles

• As $|x|/W \to \infty$ then $xL \to L_0$ and $xF \to |x|F_0$.

• $L(\psi)$ corresponds to *localized* velocity profile. $F(\psi)$ corresponds to *non-localized* profile. Require localized profile, so $F_0 = 0$.

• Velocity profiles outside separatrix (using b.c. on $n$):

\[
V_{yi} \simeq + \frac{\mu_e}{\mu_i + \mu_e} \frac{|x|}{\langle x^2 \rangle};
\]

\[
V_{yE \times B} \simeq - \frac{\left( \mu_i \tau - \mu_e \right)}{(1 + \tau)(\mu_i + \mu_e)} \frac{|x|}{\langle x^2 \rangle};
\]

\[
V_{ye} = - \frac{\mu_i}{\mu_i + \mu_e} \frac{|x|}{\langle x^2 \rangle}.
\]
Island Propagation

• As $|\chi|/W \to \infty$ expect $V_y E \times B \to V_{EB} - V$, where $V_{EB}$ is unperturbed (i.e., no island) $\vec{E} \times \vec{B}$ velocity at rational surface (in lab. frame), and $V$ is island phase-velocity (in lab. frame).

• Hence

$$V = V_{EB} + \frac{(\mu_i \tau - \mu_e)}{(1 + \tau) (\mu_i + \mu_e)}.$$  

• But unperturbed ion/electron fluid velocities (in lab. frame):

$$V_i = V_{EB} + \tau/(1 + \tau), \quad V_e = V_{EB} - 1/(1 + \tau).$$

• Hence

$$V = \frac{\mu_i}{\mu_i + \mu_e} V_i + \frac{\mu_e}{\mu_i + \mu_e} V_e.$$  

So, island phase-velocity is *viscosity weighted average* of unperturbed ion/electron fluid velocities.
Polarization Term - I

- Vorticity equation yields

\[ J_c \simeq \frac{1}{2} \left( x^2 - \frac{\langle x^2 \rangle}{\langle 1 \rangle} \right) \frac{d[M (M + \tau L)]}{d\psi} + I(\psi) \]

outside separatrix, where \( J_c \) is part of \( J \) with \( \cos \theta \) symmetry.

- As before, flux-surface average of Ohm’s law yields:

\[ \langle J_c \rangle = I(\psi) \langle 1 \rangle = \eta^{-1} \frac{d\Psi}{dt} \langle \cos \theta \rangle. \]

- Hence

\[ J_c \simeq \frac{1}{2} \left( x^2 - \frac{\langle x^2 \rangle}{\langle 1 \rangle} \right) \frac{d[M (M + \tau L)]}{d\psi} + \eta^{-1} \frac{d\Psi}{dt} \langle \cos \theta \rangle \]
Polarization Term - II

- Asymptotic matching between inner and outer regions yields:

\[ \Delta' \Psi = -4 \int_{-\infty}^{+\Psi} \langle J_c \cos \theta \rangle d\psi. \]

- Evaluating flux-surface integrals, making use of previous solutions for \( M \) and \( L \), obtain modified Rutherford equation:

\[
\frac{0.823}{\eta} \frac{dW}{dt} \simeq \Delta' + 5.52 \beta (V - V_{EB}) (V - V_i) \frac{L_s^2}{L_n^2} \frac{1}{W^3}.
\]

- New term is due to \textit{polarization current} associated with ion fluid flow around curved island flux-surfaces (in island frame). Obviously, new term is zero if island propagates with ion fluid: \textit{i.e.}, \( V = V_i \).
Main Predictions of Subsonic Island Theory

• Results limited to large islands: i.e., large enough for sound waves to flatten density profile.

• Island propagates at (perpendicular) viscosity weighted average of unperturbed (no island) ion and electron fluid velocities.

• Polarization term in Rutherford equation is stabilizing provided ion (perpendicular) viscosity greatly exceeds electron (perpendicular) viscosity (which is what we expect), and destabilizing otherwise.

• Polarization term \( \propto W^{-3} \) dominates bootstrap term \( \propto W^{-1} \) at small island widths, and vice versa at large island widths. Thus, polarization term can also provide threshold effect that prevents NTMs from growing until they exceed critical island width.
8: Supersonic Island Theory
Drift-MHD Equations\textsuperscript{a}

- Steady-state drift-MHD equations (with $\tau = 0$, since ion diamagnetic effects largely irrelevant to supersonic islands):

\[
\begin{align*}
\psi &= -\frac{x^2}{2} + \Psi \cos \theta, \quad U = \nabla^2 \phi, \\
0 &= [\phi - n, \psi] + \eta J, \\
0 &= [\phi, U] + [J, \psi] + \mu_i \nabla^4 \phi, \\
0 &= [\phi, n] + [V_z + J, \psi] + D \nabla^2 n, \\
0 &= [\phi, V_z] + \alpha [n, \psi] + \mu_i \nabla^2 V_z.
\end{align*}
\]

Zero-\(\alpha\) Solution

- By definition, highlighted term small for supersonic islands.
- If term completely neglected, obtain trivial solution:
  \[ \phi = n = -x, \quad U = V_z = J = 0. \]
- Island propagates with electron fluid.
- Island does not perturb ion fluid, so zero polarization current.
**Small-\(\alpha\) Solution**

- Assume that highlighted term small, but not negligible. Perturb about zero-\(\alpha\) solution.

- So

\[
\phi = -x + \delta \phi, \quad n = -x + \delta n,
\]

where \(\delta \phi, \delta n, U, V_z, J\) all \(O(\alpha) \ll 1\).
Analysis - I

• Lowest order solution:

\[
\delta n = \delta \phi + H(\psi),
\]
\[
J = -\tilde{G} + (\alpha/2) \tilde{x}^2,
\]
\[
V_z = -\alpha (W/4)^2 \cos \theta,
\]

where \( \tilde{A} \equiv A - \langle A \rangle / \langle 1 \rangle \).

• Here, \( G = -x H' \). Now, \( G = 0 \) inside separatix, but outside separatrix

\[
G = |x| \left( \frac{\langle x \nu \rangle + \alpha (W/4)^4}{\langle x^2 \rangle} \right),
\]

where \( \nu = -\delta \phi_x \).
Analysis - II

- Perturbed velocity $v$ satisfies
  \[ v_{xx} = (D/\mu) (\overline{v} - \overline{G}) - (G - \overline{G}) - \alpha (W/4)^2 \cos \theta, \]
  where $\overline{\cdots}$ denotes a $\theta$-average at constant $x$.

- Boundary conditions: $v_x = 0$ at $x = 0$, and
  \[ v \to v_i + v'_i |x| - (\alpha/2) (W/4)^2 x^2 \cos \theta \]
  as $|x| \to \infty$.

- Above equation highly nonlinear, but can be solved via iteration.
Need for Intermediate Layer

- Inner region island solution does not satisfy $J \to 0$ as $|x| \to \infty$: *i.e., it does not asymptote to ideal-MHD solution in outer region.*
- Require *intermediate layer* between island and outer region to allow proper matching.
- Intermediate layer much wider than island, so governed by *linear physics.*
Intermediate Layer - I

- Write
  \[ \phi(x, \theta) = -x + \delta \phi(x) + \tilde{\phi}(x) e^{i \theta}. \]

- Neglect all transport terms except ion viscosity.

- Linearized drift-MHD equations yield
  \[
  \tilde{\phi}_{xx} - \tilde{v}_{xx} \tilde{\phi} - \left( \tilde{v} - \frac{\alpha x^2}{1 - i \mu_i \alpha x^2} \right) \tilde{\phi}
  = - \left( \tilde{v} - \frac{\alpha x^2}{1 - i \mu_i \alpha x^2} \right) \frac{(W/4)^2}{x},
  \]
  where \( \tilde{v} = -\delta \phi_x \).
Intermediate Layer - II

- Mean velocity profile determined by quasi-linear force balance:

\[
\overline{v}_{xx} = \frac{1}{2} \frac{\alpha^2 x^2}{1 + (\mu_i \alpha x^2)^2} \left| \left( \frac{W}{4} \right)^2 - x \ddot{\phi} \right|^2.
\]

- Perturbed current:

\[
\tilde{J} = (\ddot{\phi}_{xx} - \overline{v}_{xx} \ddot{\phi})/x.
\]
Intermediate Layer - III

• Boundary conditions as $x \to 0$:

$$
\begin{align*}
\dot{\phi} & \to 0, \\
\dot{v} & \to v_i + v_i' |x|.
\end{align*}
$$

• Boundary conditions as $|x| \to \infty$:

$$
\begin{align*}
\dot{\phi} & \to \frac{(W/4)^2}{x}, \\
\dot{v} & \to v_\infty + v_\infty' |x|.
\end{align*}
$$

• Large-$|x|$ boundary conditions ensure that $\ddot{J} \to 0$. So solution matches to ideal-MHD solution.
Physics of Intermediate Layer

- Island launches *drift-acoustic waves* into intermediate layer.
- Waves are *absorbed* in layer (due to ion viscosity).
- Waves carry *momentum*.
- Momentum exchange between island and intermediate layer ensures that velocity gradient, $v'_i$, at inner boundary of layer not same as gradient, $v'_\infty$, at outer boundary.
- For isolated island solution, require $v'_\infty = 0$. This boundary condition *uniquely specifies* solution for given values of $\alpha$, $\mu_i$, $D$, etc.
Velocity in Intermediate Layer

\[ \hat{v} \]

\[ \hat{x} \]
Current in Intermediate Layer

\[ \tilde{J} \]

\[ 0 \]

\[ 2 \times 10^{-4} \]

\[ -2 \times 10^{-4} \]

\[ \hat{x} \]
Island Propagation

- Island propagation velocity:

\[ V = V_e - 0.27 \left( \frac{W}{4} \right)^3 \alpha^{3/4} D^{-1} - 0.24 \left( \frac{W}{4} \right)^4 \alpha^{1/3} \mu^{-4/3}. \]

- Island phase velocity close to unperturbed electron fluid velocity, but dragged slightly in ion direction due to sound-wave effects.
Ion Polarization Term

- Rutherford Equation:

\[
\frac{0.823}{\eta} \frac{dW}{dt} = \Delta' - \frac{\beta}{\alpha^{1/4}} \frac{L_s^2}{L_n^2} [1.5 + 0.38 (W/4)^2 D^{-1}].
\]

- Sound-wave effects ensure ion fluid slightly perturbed by island, generating polarization term in Rutherford equation. Term is *stabilizing*. 

Maximum Island Width

- Supersonic branch of solutions ceases to exist beyond maximum island width:

\[ W_{\text{max}} = 0.36 \alpha^{-1/12} D^{1/3}. \]

- Hypothesized that island bifurcates to subsonic solution branch when \( W > W_{\text{max}} \). This type of behavior has been observed in computer simulations.\(^a\)

Main Predictions of Supersonic Island Theory

• Results limited to small islands: *i.e.*, small enough that sound waves cannot flatten density profile.

• Islands phase velocities close to unperturbed electron fluid velocity, but dragged slightly in ion direction by sound wave effects.

• Islands radiate drift-acoustic waves.

• Momentum carried by drift-acoustic waves gives rise to strong velocity shear in region surrounding islands.

• Polarization term in Rutherford island equation is stabilizing.

• Supersonic branch ceases to exist above critical island width.
9: Other Effects
Neoclassical Flow Damping

- Poloidal (and, sometimes, toroidal) flow strongly damped in low-collisionality plasmas typically found in tokamaks.
- Flow damping affects island propagation velocity, which modifies ion polarization term in Rutherford equation.
- Require drift-MHD island theory that takes flow damping into account.\(^a\)

Finite Trapped Ion Orbit Width

- Width of trapped ion orbit of order

$$\rho_\theta = \left( \frac{B_z}{B_\theta} \right) \rho.$$  

- In conventional tokamak, trapped ion orbit width often comparable with island width.

- Kinetic analysis required to take finite orbit widths into account.\(^a\)

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\(^a\) A. Bergmann, E. Poli, and A.G. Peeters, Phys. Plasmas 12, 072501 (2005).
Magnetic Field-Line Curvature

• Magnetic field-line curvature in tokamak plasmas gives rise to particle drifts that are three-dimensional in nature, and cannot be captured in two-dimensional slab model.

• Three-dimensional island theory required to take curvature drifts into account.\(^a\)

Drift-Wave Turbulence

- Perpendicular transport that determines island profiles actually due to drift-wave turbulence.
- Radial extent of drift-wave eddies of order $\rho$. Hence, eddies can easily be comparable in width to island.
- Island theory in which island immersed in bath of drift-wave turbulence required when eddy width comparable with island width. Turbulence affects island by modifying island profiles. Island profiles affect drift-wave stability, and hence turbulence levels. Theory must self-consistently determine effect of turbulence on island, and effect of island on turbulence.\(^a\)