Stabilization of External Kink Modes in Magnetic Fusion Experiments Using a Thin Conducting Shell

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Abstract

In nearly all magnetic fusion devices the plasma is surrounded by a conducting shell of some description. In most cases this is the vacuum vessel. What effect does a conducting shell have on the stability of external kink modes? Is there any major difference between the effect of a perfectly conducting shell and a shell of finite conductivity? What happens if the shell is incomplete? These, and other, questions are explored in detail in this lecture using simple resistive magnetohydrodynamical (resistive MHD) arguments. Although the lecture concentrates on one particular type of magnetic fusion device, namely, the tokamak, the analysis is fairly general and could also be used to examine the effect of conducting shells on other types of device (e.g. Reversed Field Pinches, Stellerators, etc.).

1 Introduction

In the earliest tokamaks the plasma was surrounded by a close fitting, thick conducting shell whose L/R time was much longer than the typical pulse length of the discharge. Image currents in the shell held the plasma equilibrium in place without the need for feedback and also stabilized external kink modes. The shell contained an insulating break at at least one toroidal location in order to allow the poloidal magnetic field to diffuse into the plasma. There are a number of disadvantages associated with a thick shell. First, a thick shell is obviously not relevant to the final goal of fusion research, which is a steady state reactor. Second, it is very difficult to diagnose a plasma which is encased in inch thick metal. Finally, the insulating breaks in a thick shell gives rise to induced error fields whenever the poloidal magnetic field is changed. Error fields can cause problems in tokamak plasmas. Modern tokamaks have generally dispensed with thick shells and instead employ thin shells whose L/R times are significantly less than the pulse length. These shells are not necessarily close fitting and are often incomplete. The use of a thin shell necessitates feedback control of the horizontal (and vertical) position of the plasma. But, what effect does a thin shell have on the stability of external kink modes?

2 Newcomb's Criterion

External kink modes are fast growing MHD instabilities whose growth rate, γ, is moderated by plasma inertia. Plasma viscosity and resistivity play no significant role for these modes. The linear stability problem can be posed as a real second order differential equation for the plasma displacement in which inertia appears as a γ^2 term. Physical boundary conditions must be satisfied at the magnetic axis and also far from the plasma. The displacement function is undetermined to an arbitrary multiplicative constant (since this is a linear problem). An appropriate choice of γ^2 allows both boundary conditions to be satisfied simultaneously. If $\gamma^2 < 0$ then two purely oscillatory modes are obtained. These modes are part of the Alfvén wave continuum and are damped by various kinetic effects (e.g. Landau damping) which are not taken into account in magnetohydrodynamics. If $\gamma^2 > 0$

then a growing and a decaying mode are obtained. The growing mode (which grows on the characteristic hydromagnetic timescale of the plasma) corresponds to the external kink mode.

Newcomb's criterion is a neat trick by which one can discover whether or not the external kink mode is unstable in the absence of a conducting shell using just the *marginally stable* equations of ideal MHD; i.e. by neglecting plasma inertia, as well as plasma viscosity and resistivity. Newcomb's criterion does not yield any information about the growth rates of unstable modes. This is generally not a problem since external kink modes are such fast growing modes that their growth rates are of only academic interest (i.e. it does not really matter to an experimentalist whether the mode destroys the plasma in one microsecond or five microseconds; to all intents and purposes a plasma which is unstable to an external kink mode is useless). Newcomb's criterion is easy to explain but is surprisingly difficult to prove mathematically. Hence, a mathematical proof in not attempted in this lecture.

Consider a large aspect ratio, low β, tokamak plasma which is approximated, in the usual manner, as a periodic cylinder. Conventional cylindrical polar coordinates (r, θ, ϕ) are adopted. It is convenient to define a simulated toroidal angle $\phi = z/R_0$, where R_0 is the simulated major radius of the plasma. The equilibrium magnetic field is written $\mathbf{B} = (0, B_\theta(r), B_\phi)$, and the associated plasma current takes the form $\mathbf{j} = (0, 0, j_\phi(r)\,)$, where

$$\mu_0 j_\phi(r) = \frac{1}{r}\frac{d(rB_\theta)}{dr}. \tag{1}$$

Equilibrium magnetic field lines satisfy the differential equation

$$\frac{d\phi}{d\theta} = q(r), \tag{2}$$

where the "safety factor"

$$q(r) = \frac{rB_\phi}{R_0 B_\theta} \tag{3}$$

parameterizes the helical pitch of the field lines. In a conventional tokamak plasma the safety factor is a monotonically increasing function of the flux surface radius r. Furthermore, $q \sim O(1)$.

Consider the stability of an external kink mode with m periods in the poloidal direction and n periods in the poloidal direction. The perturbed magnetic field and the perturbed plasma current can be written in terms of a flux function:

$$\delta\mathbf{B} = \nabla\psi\wedge\hat{\mathbf{z}}, \tag{4a}$$

$$\mu_0\delta\mathbf{j} = \nabla\wedge\delta\mathbf{B} = -\nabla^2\psi\,\hat{\mathbf{z}}, \tag{4b}$$

where

$$\psi(r,\theta,\phi,t) = \psi(r)\exp[\mathrm{i}\,(m\theta - n\phi)] \tag{5}$$

for a marginally stable mode. The linearized, marginally stable, ideal MHD force balance equation takes the form

$$-\nabla\delta p + \delta\mathbf{j}\wedge\mathbf{B} + \mathbf{j}\wedge\delta\mathbf{B} = \mathbf{0}, \tag{6}$$

where δp is the perturbed plasma pressure. The curl of the above relation yields the "cylindrical tearing mode equation",

$$\nabla^2\psi + \frac{\mu_0 j_\phi'}{B_\theta(nq/m-1)}\,\psi = 0, \tag{7}$$

where $j_\phi' \equiv dj_\phi/dr$, and

$$\nabla^2\psi \simeq \frac{1}{r}\frac{d}{dr}\left(r\,\frac{d\psi}{dr}\right) - \frac{m^2}{r^2}\,\psi \tag{8}$$

in the large aspect ratio limit.

Suppose that a marginally stable test solution ψ (i.e. a solution to Eq. (7)) is launched from the magnetic axis and integrated past the edge of the plasma ($r=a$, say) out to large r. The launch conditions are adjusted so that the solution is well behaved close to the magnetic axis. The physical boundary condition to be satisfied at large r is $\psi(r)\to 0$ as $r\to\infty$. Newcomb's criterion hinges on the following very simple observation. Suppose that, instead of Eq. (7), a differential equation which takes plasma inertia into account is used to evolve ψ out to large r. The function $\psi(r)$ curls over move than the marginally stable solution if $\gamma^2 < 0$ and less than the marginally stable solution if $\gamma^2 > 0$. This is illustrated in Fig. 1.

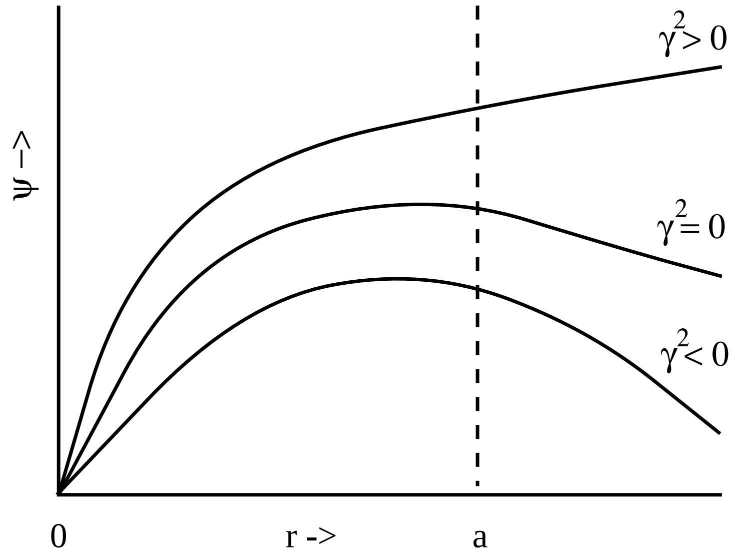

Figure 1: A schematic diagram showing the relationship between unstable, oscillatory, and marginally stable test solutions.

Suppose that the marginally stable test solution never crosses the axis ($\psi = 0$) as $r \to \infty$ but instead blows up to $+\infty$. This solution clearly does not satisfy the correct boundary condition at large r. However, according to Fig. 1, a judicious choice of $\gamma^2 < 0$ can cause an oscillatory test solution to curl over more than the marginally stable solution so that it is bounded as $r \to \infty$. This is illustrated in Fig. 2. Suppose that the marginally stable test solution crosses the axis at finite r and blows up to $-\infty$. This solution also does not satisfy the correct boundary condition at large r. However, according to Fig. 1, a judicious choice of $\gamma^2 > 0$ can cause a growing or decaying test solution to curl over less than the marginally stable solution so that it is bounded as $r \to \infty$. This is illustrated in Fig. 3. Thus, Newcomb's criterion can be expressed as follows:

> If a marginally stable test solution, launched from the magnetic axis and integrated to large r, changes sign before reaching $r = \infty$ then the external kink mode is unstable. Otherwise, it is stable.

The test solution $\psi(r)$ is completely specified by two parameters; its amplitude

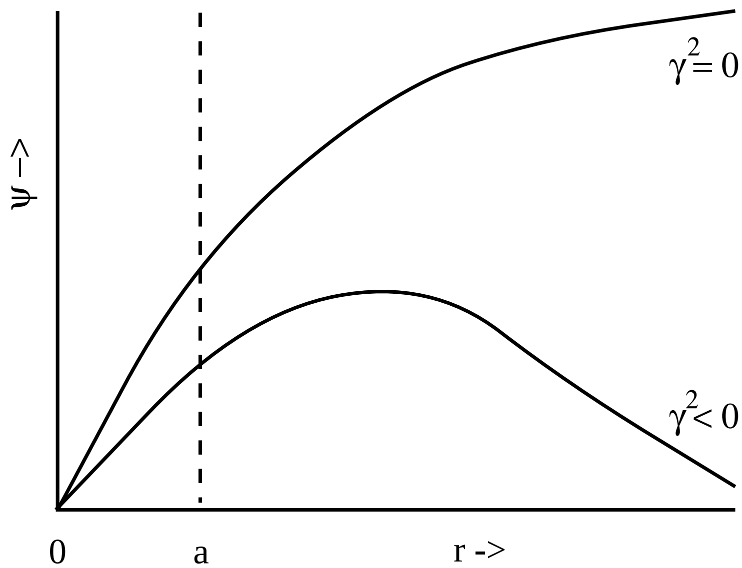

Figure 2: A schematic diagram showing how the large r boundary condition is satisfied when the marginally stable test solution does not cross the axis.

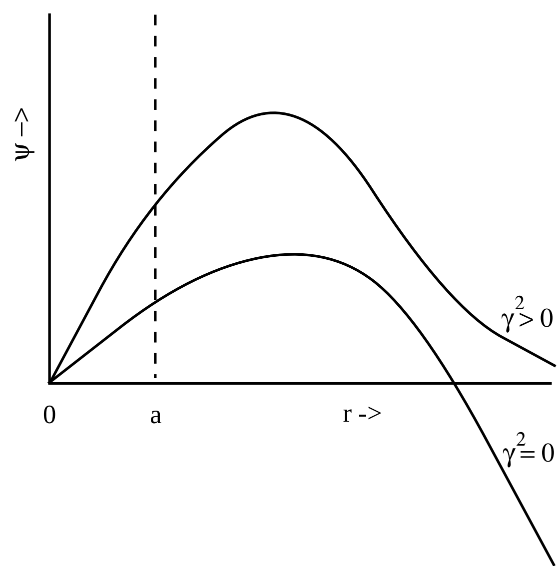

Figure 3: A schematic diagram showing how the large r boundary condition is satisfied when the marginally stable test solution crosses the axis.

$\Psi_a \equiv \psi(a)$ at the edge of the plasma, and

$$\lambda = -\frac{a}{m\Psi_a}\left.\frac{d\psi}{dr}\right|_{r=a}. \tag{9}$$

The former parameter is arbitrary in a linear problem, so the stability of the external kink mode is completely determined by the single parameter λ. In the vacuum region outside the plasma (i.e. $r > a$) the ideal MHD force balance equation, (7), yields $\nabla^2\psi = 0$. It follows that $\psi(r)$ is a linear combination of r^{+m} and r^{-m} functions in this region. It is easily demonstrated that

$$\psi(r > a) = \frac{\Psi_a}{2}\left((1-\lambda)\left(\frac{r}{a}\right)^m + (1+\lambda)\left(\frac{r}{a}\right)^{-m}\right). \tag{10}$$

Another way of expressing Newcomb's criterion is that the external kink mode is unstable if $\psi(r \to \infty)/\Psi_a < 0$. It follows from Eq. (10) that the criterion for instability is

$$\lambda > 1. \tag{11}$$

Equation (7) can be integrated to give

$$\lambda = \int_0^1 \left[\frac{\mu_0 r(-j_\phi')}{B_\theta(m - nq)} - \frac{m}{r}\right]\psi\, dr, \tag{12}$$

where use has been made of $\psi \propto r^m$ as $r \to 0$. In the above formula, all lengths are normalized to the minor radius, a, and $\psi(r)$ is normalized to unity at $r = a$. In a conventional tokamak plasma $j_\phi' < 0$ and $B_\theta > 0$. Moreover, the m/n external kink mode is only relevant when the rational surface, defined $q(r_s) = m/n$, lies outside the plasma. When the rational surface moves inside the plasma the m/n external kink modes converts into a tearing mode. It follows that $m > nq$ in the above formula. Clearly, the first term in Eq. (12), which involves the current gradient, is destabilizing, whereas the second term is stabilizing. An external kink mode becomes more unstable as its rational surface approaches the edge of the plasma (i.e. $m - nq \to 0$) but becomes more stable as its poloidal mode number increases.

3 Effect of an Ideal Shell

Suppose that the plasma is surrounded by a perfectly conducting shell whose inner radius is r_w (where $r_w > a$). How is the stability of the external kink mode affected? The physical boundary condition at the shell is $\psi(r_w) = 0$. Similar arguments to those employed above yield a modified form of Newcomb's criterion:

> If a marginally stable test solution, launched from the magnetic axis and integrated to large r, changes sign before encountering an ideal shell then the external kink mode is unstable. Otherwise, it is stable.

This rule can be summed up as the requirement that $\psi(r_w)/\Psi_a < 0$ for instability. According to Eq. (10), the m/n external kink mode is only unstable in the presence of an ideal shell provided

$$\lambda > \lambda_c \equiv \frac{1 + (a/r_w)^{2m}}{1 - (a/r_w)^{2m}}. \tag{13}$$

This is a more onerous criterion to satisfy than Eq. (11). In particular, as the inner radius of the shell approaches the plasma radius the critical value of λ above which the external kink mode is unstable tends to infinity. Since λ is finite, according to Eq. (12), this implies that complete stabilization of the external kink mode is achieved by placing an ideal shell right at the edge of the plasma. For a given plasma equilibrium (with a given value of λ, which can be calculated) the external kink mode is stabilized if an ideal shell is placed sufficiently close to the plasma. The exact criterion for stabilization is

$$\frac{r_w}{a} < \frac{r_c}{a} \equiv \left(\frac{\lambda + 1}{\lambda - 1}\right)^{1/2m}, \tag{14}$$

assuming that $\lambda > 1$ (i.e. the external kink mode is unstable in the absence of a shell).

4 Effect of a Thin Resistive Shell

There is, of course, no such thing as a perfectly conducting shell. However, a thick shell may be approximated as a perfect conductor if its L/R time is much longer than the typical pulse length. Suppose that this is not the case. How does a thin shell whose L/R time is much less than the pulse length affect the stability of external kink modes? The L/R time, or “time constant”, of the shell is defined

$$\tau_w = \mu_0 \sigma_w r_w \delta_w, \tag{15}$$

where σ_w, r_w, and δ_w are the shell conductivity, radius, and thickness, respectively. In the “thin shell” limit, which corresponds to

$$\frac{\delta_w}{r_w} \ll |\gamma| \tau_w \ll \frac{r_w}{\delta_w}, \tag{16}$$

where γ is the growth rate, the skin depth in the material which makes up the shell is much less than its radius but much greater than its thickness. In this regime there is negligible radial variation of the magnetic flux function $\psi(r)$ across the shell. In addition, Ohm’s law and Faraday’s law integrated across the shell yield

$$\left[r \frac{d\psi}{dr} \right]_{r_{w-}}^{r_{w+}} = \gamma \tau_w \, \Psi_w, \tag{17}$$

where $\Psi_w \equiv \psi(r_w)$ is the magnetic flux which penetrates the shell.

Consider a mode which grows on the L/R time of the shell. Such a mode is marginally stable as far as ideal MHD is concerned, so $\psi(r)$ satisfies Eq. (7) everywhere apart from inside the shell. A solution which satisfies all of the boundary conditions can be constructed by using the solution (10) in the region $a < r < r_w$, and using

$$\psi(r) = \Psi_w \left(\frac{r}{r_w} \right)^{-m} \tag{18}$$

in the region $r > r_w$. This solution is sketched in Fig. 4. It is helpful to define the “shell stability index”,

$$\Delta_w = \left[r \frac{d\psi}{dr} \Big/ \psi \right]_{r_{w-}}^{r_{w+}}. \tag{19}$$

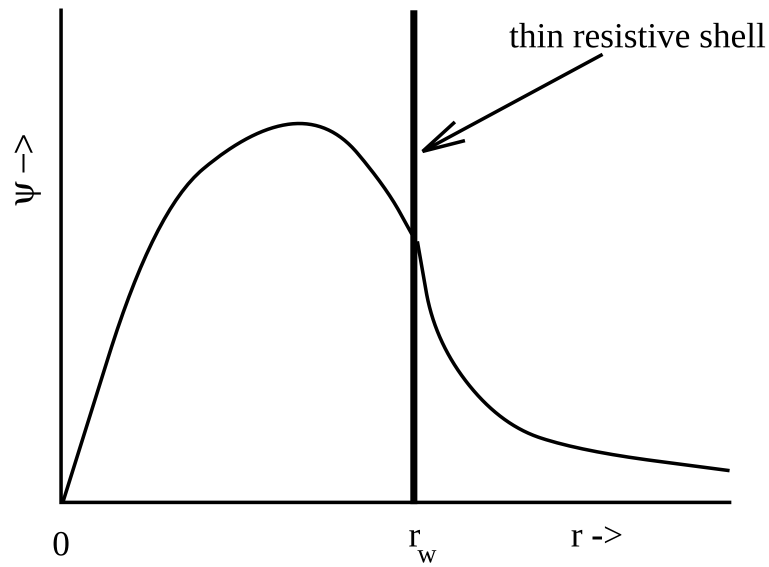

Figure 4: A schematic diagram showing a typical resistive shell mode solution.

It is clear from Eqs. (10) and (18) that

$$\Delta_w = \frac{2m\,(\lambda - 1)}{(1+\lambda)(a/r_w)^{2m} - (\lambda - 1)}. \tag{20}$$

A comparison of Eqs. (17) and (19) yields

$$\gamma \tau_w = \Delta_w. \tag{21}$$

According to the above formula, a non rotating mode which grows on the L/R time of the shell is unstable whenever $\Delta_w > 0$. This mode is generally termed the "resistive shell mode". Equation (20) implies that the resistive shell mode is only unstable when $\lambda > 1$; i.e. whenever the ideal external kink mode is unstable in the absence of a shell. The growth rate of the resistive mode becomes infinite when $\lambda = \lambda_c$ (see Eq. (13)) and the mode is stable for $\lambda > \lambda_c$. $\lambda = \lambda_c$ corresponds to the marginal stability criterion for the ideal mode. The resistive shell mode is unstable for $1 < \lambda < \lambda_c$ and the ideal mode is unstable for $\lambda_c < \lambda$. Thus, either a resistive or an ideal mode is unstable whenever $\lambda > 1$.

Clearly, if the ideal external kink mode is unstable in the absence of a shell (i.e. if $\lambda > 1$) then either the resitive shell mode or the ideal kink mode are unstable in the presence of a resistive shell. In other words, a resistive shell does not improve the stability of a tokamak plasma against external modes. In marked contrast, an ideal shell can completely stabilize external modes if it is placed sufficiently close to the plasma.

For a given plasma equilibrium, with a given value of λ ($\lambda > 1$), the resistive shell mode is unstable when the shell lies too close to the plasma; i.e. when $a < r < r_c$ (see Eq. (14)). On the other hand, the ideal external kink mode is unstable when the shell lies too far from the plasma; i.e. when $r > r_c$. In the simplest theory there is no position of the shell for which one or the other of these modes is not unstable. Recently, it has been established that if the plasma is rapidly rotating then a narrow window of stability opens up for shell radii just less than r_c. If the shell radius lies in the stability window then neither the resistive shell mode nor the ideal kink mode are unstable.

5 Effect of a Partial Shell

What effect does a partial shell have on the stability of external kink modes? It is helpful to write the resistive shell mode solution in the form (see Fig. 5)

$$\psi(r) = \left(1 + \frac{\gamma\tau_w}{2m}\right)\Psi_w\,\hat{\psi}_{\rm plasma}(r) - \frac{\gamma\tau_w}{2m}\Psi_w\,\hat{\psi}_{\rm shell}(r). \tag{22}$$

Here, $\hat{\psi}_{\rm plasma}(r)$ is that part of the solution which is maintained by plasma currents and $\hat{\psi}_{\rm shell}(r)$ is that part which is maintained by eddy currents induced in the shell. Both, $\hat{\psi}_{\rm plasma}$ and $\hat{\psi}_{\rm shell}$ are normalized to unity at the shell radius. It is easily demonstrated that

$$\left[r\,\frac{d\hat{\psi}_{\rm shell}}{dr}\right]_{r_{w-}}^{r_{w+}} = -2m, \tag{23}$$

whereas

$$\left[r\,\frac{d\hat{\psi}_{\rm plasma}}{dr}\right]_{r_{w-}}^{r_{w+}} = 0. \tag{24}$$

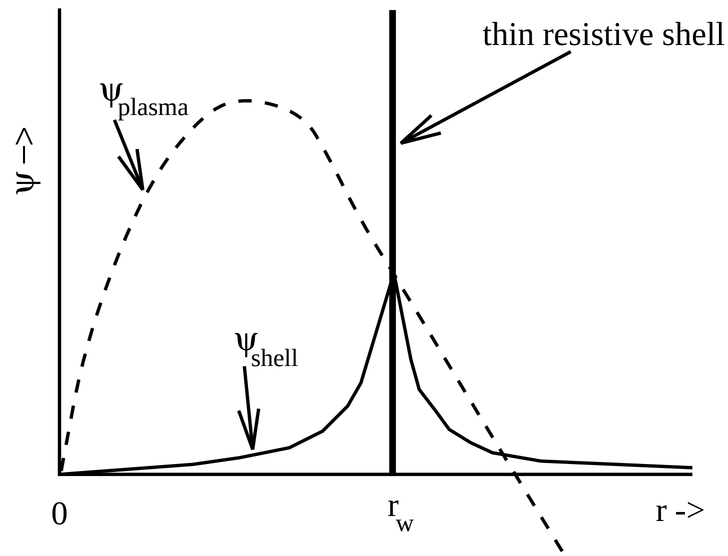

Figure 5: A schematic diagram showing the plasma and shell solutions.

It follows from Eqs. (22)–(24) that

$$\int_{r_{w-}}^{r_{w+}} \delta j_\phi \, dr = -\frac{1}{\mu_0}\left[\frac{d\psi}{dr}\right]_{r_{w-}}^{r_{w+}} = -\frac{\gamma\tau_w \Psi_w}{\mu_0 r_w} = -\frac{\partial}{\partial t}\int_{r_{w-}}^{r_{w+}} \sigma_w \psi \, dr, \qquad (25)$$

as demanded by Ohm's law ($\delta j_\phi = -\sigma_w \partial\psi/\partial t$).

According to Eqs. (5) and (22) the perturbed poloidal flux in the vicinity of a complete shell is given by

$$\begin{aligned}\psi(r,\theta,\phi) &= \left[\left(1+\frac{\gamma\tau_w}{2m}\right)\Psi_w\,\hat{\psi}_{\rm plasma}(r)\right.\\ &\qquad \left. -\frac{\gamma\tau_w}{2m}\Psi_w\,\hat{\psi}_{\rm shell}(r)\right]\exp({\rm i}\,(m\theta-n\phi)\,). \qquad (26)\end{aligned}$$

A partial shell consists of conducting metal at some angular coordinates, (θ, ϕ), and vacuum gaps at the remaining coordinates. Suppose that Eq. (26) still holds at angular coordinates corresponding to metal sections of the shell, but at coordinates corresponding to vacuum gaps

$$\psi(r,\theta,\phi) = \left[\left(1+\frac{\gamma\tau_w}{2m}\right)\Psi_w\,\hat{\psi}_{\rm plasma}(r)\right]\exp({\rm i}\,(m\theta-n\phi)\,). \qquad (27)$$

Equation (27) is the same as Eq. (26) except that the part of the perturbed poloidal flux which is generated by eddy currents flowing in the shell is missing (since there are no eddy currents flowing in the vacuum gaps). The m/n harmonic of the perturbed poloidal flux is given by

$$\psi^{m/n}(r) = \oint\oint \psi(r,\theta,\phi)\, \exp[-{\rm i}\,(m\theta - n\phi)]\, \frac{d\theta}{2\pi}\frac{d\phi}{2\pi}. \tag{28}$$

It follows from Eqs. (26) and (27) that

$$\psi^{m/n}(r) = \left(1+\frac{\gamma\tau_w}{2m}\right)\Psi_w\,\hat{\psi}_{\rm plasma}(r) - (1-f)\,\frac{\gamma\tau_w}{2m}\,\Psi_w\,\hat{\psi}_{\rm shell}(r), \tag{29}$$

where f is the area fraction of vacuum gaps in the shell.

The shell stability index for the m/n mode satisfies

$$\Delta_w = \left[r\,\frac{d\psi^{m/n}}{dr}\bigg/\psi^{m/n}\right]_{r_{w-}}^{r_{w+}}. \tag{30}$$

It follows from Eqs. (23), (24), and (29), that

$$\Delta_w = \frac{(1-f)\,\gamma\tau_w}{1+f\,\gamma\tau_w/2m}. \tag{31}$$

This can be rearranged to give

$$\gamma\tilde{\tau}_w = \frac{\Delta_w}{1-\Delta_w/\Delta_c}, \tag{32}$$

where

$$\tilde{\tau}_w = (1-f)\,\tau_w \equiv (1-f)\,\mu_0\sigma_w r_w\delta_w, \tag{33}$$

and

$$\Delta_c = 2m\left(\frac{1}{f}-1\right). \tag{34}$$

Here, $\tilde{\tau}_w$ is the time constant of a uniform shell which contains the same amount of metal as the partial shell. It is easily demonstrated from Eqs. (26) and (27)

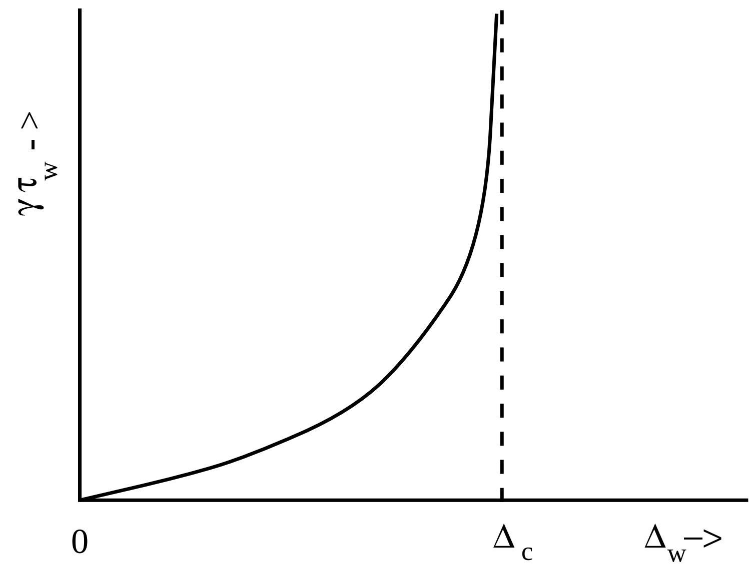

Figure 6: A schematic diagram showing the typical variation of the growth rate with the shell stability index for a partial shell.

that the ratio of the amplitude of the perturbed poloidal flux in the metal and gap sections of the shell is given by

$$\frac{\Psi_{\rm gap}}{\Psi_{\rm metal}} = 1 + \frac{\gamma\tau_w}{2m}. \tag{35}$$

The predicted (see Eq. (32)) variation of γ with Δ_w is sketched in Fig. 6. Suppose that Δ_w is gradually increased from a small positive value. Initially, the poloidal flux is evenly distributed over the metal and gap sections of the shell (see Eq. (35)) and the partial shell acts like a uniform shell containing an equal amount of metal (i.e. $\gamma\tilde{\tau}_w \simeq \Delta_w$). However, as the m/n mode becomes more unstable the poloidal flux starts to concentrate in the gap sections of the shell and the growth rate accelerates. Eventually, at a critical shell stability index, Δ_c, the poloidal flux is entirely concentrated in the gap sections of the shell and the resistive growth rate becomes infinite. It is easy to demonstrate from Newcomb's criterion that the m/n ideal external kink mode is unstable for $\Delta_w > \Delta_c$. Thus, when the shell stability index exceeds the critical value Δ_c the mode "splurges" through the gaps in the shell with an ideal growth rate.

Equation (31) can be rewritten

$$\gamma\tau_w = \hat{\Delta}_w, \tag{36}$$

where $\hat{\Delta}_w$ is the shell stability index for a shell located at radius

$$\hat{r}_w = r_w/(1-f)^{1/2m}. \tag{37}$$

Thus, a partial shell acts just like a complete shell (with the same time constant) which is located further away from the plasma. As the area fraction of gaps tends to unity the effective shell radius tends to infinity (i.e. the shell has no appreciable effect on the stability of external kink modes).

The analysis outlined in this section is somewhat heuristic, but more exact calculations reveal that it is essentially correct provided that the poloidal and toroidal extents of any gap or metal sections of the shell are much larger than the poloidal wavelength of the m/n mode, which is r_w/m. The above results also tend to become inaccurate if the shell is located very close to the plasma.

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