Magnetic Island Theory - I

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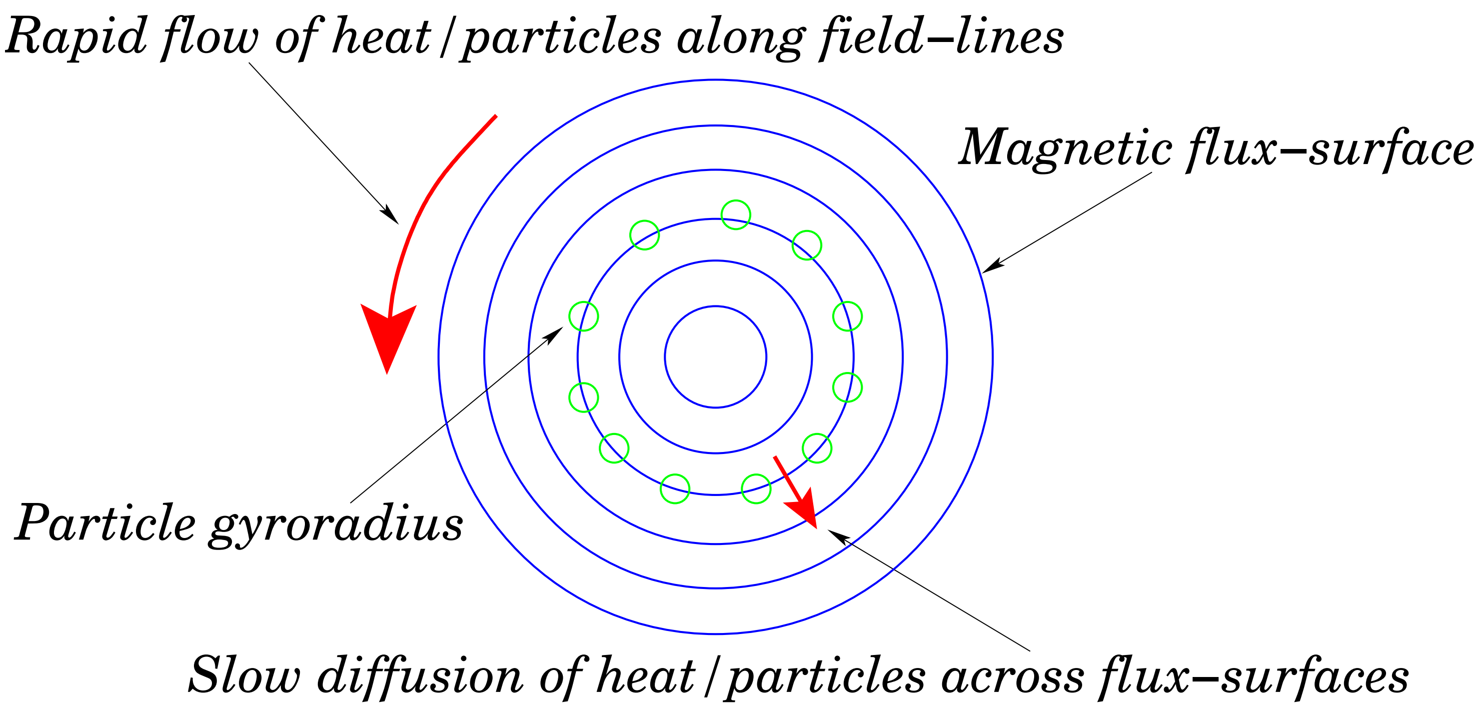
Introduction: Toroidal Magnetic Confinement

- Toroidal magnetic confinement devices designed to trap hot plasma within set of toroidally nested magnetic flux-surfaces.
- Basic principle—charged particles free to stream along field-lines, but “stick” to magnetic flux-surfaces due to (relatively) small gyroradii.
- Energy flows rapidly along field-lines, but can only diffuse across flux-surfaces relatively slowly. Diffusion rate controlled by small-scale plasma turbulence.

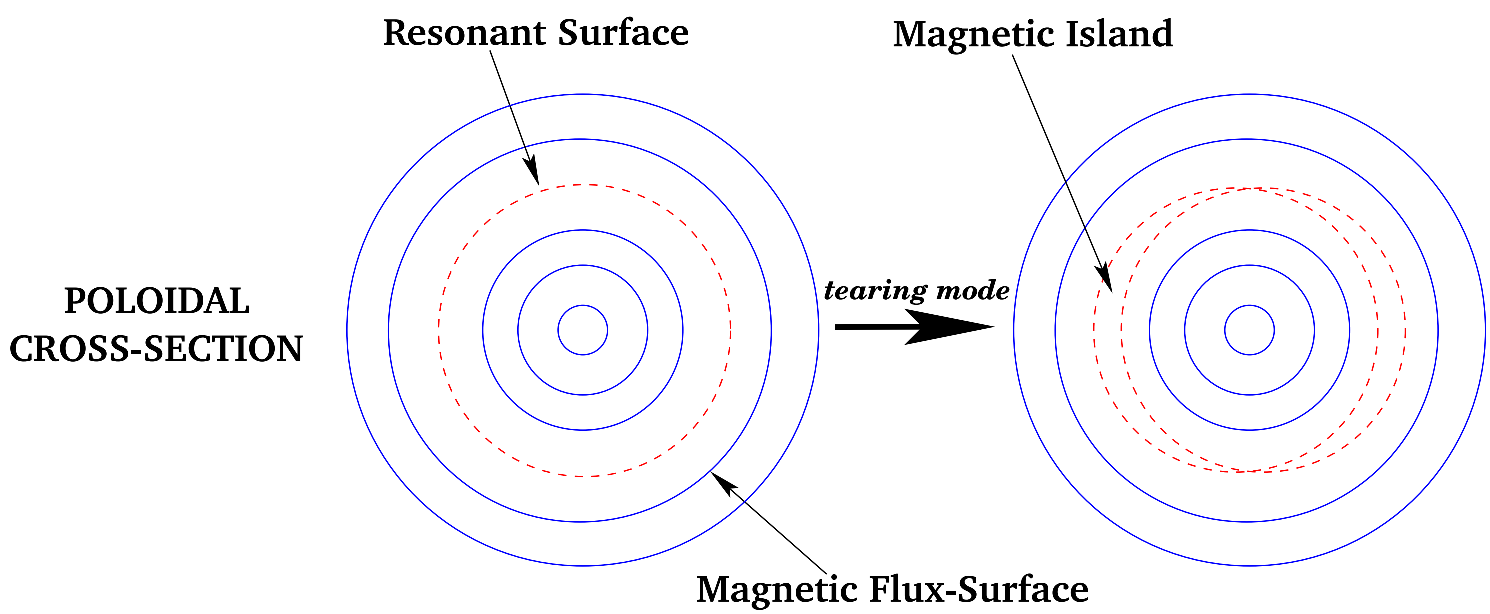
Poloidal Cross–Section of Toroidal Confinement Device



Introduction: Magnetohydrodynamical Instabilities

- Two main types of MHD instability:
  - Catastrophic "ideal" instabilities, which destroy plasma in matter of micro-seconds—we know how to avoid these.
  - Slowly growing "tearing" instabilities, which reconnect magnetic flux-surfaces, thereby degrading their confinement properties—much harder to avoid.

Introduction: Tearing Modes

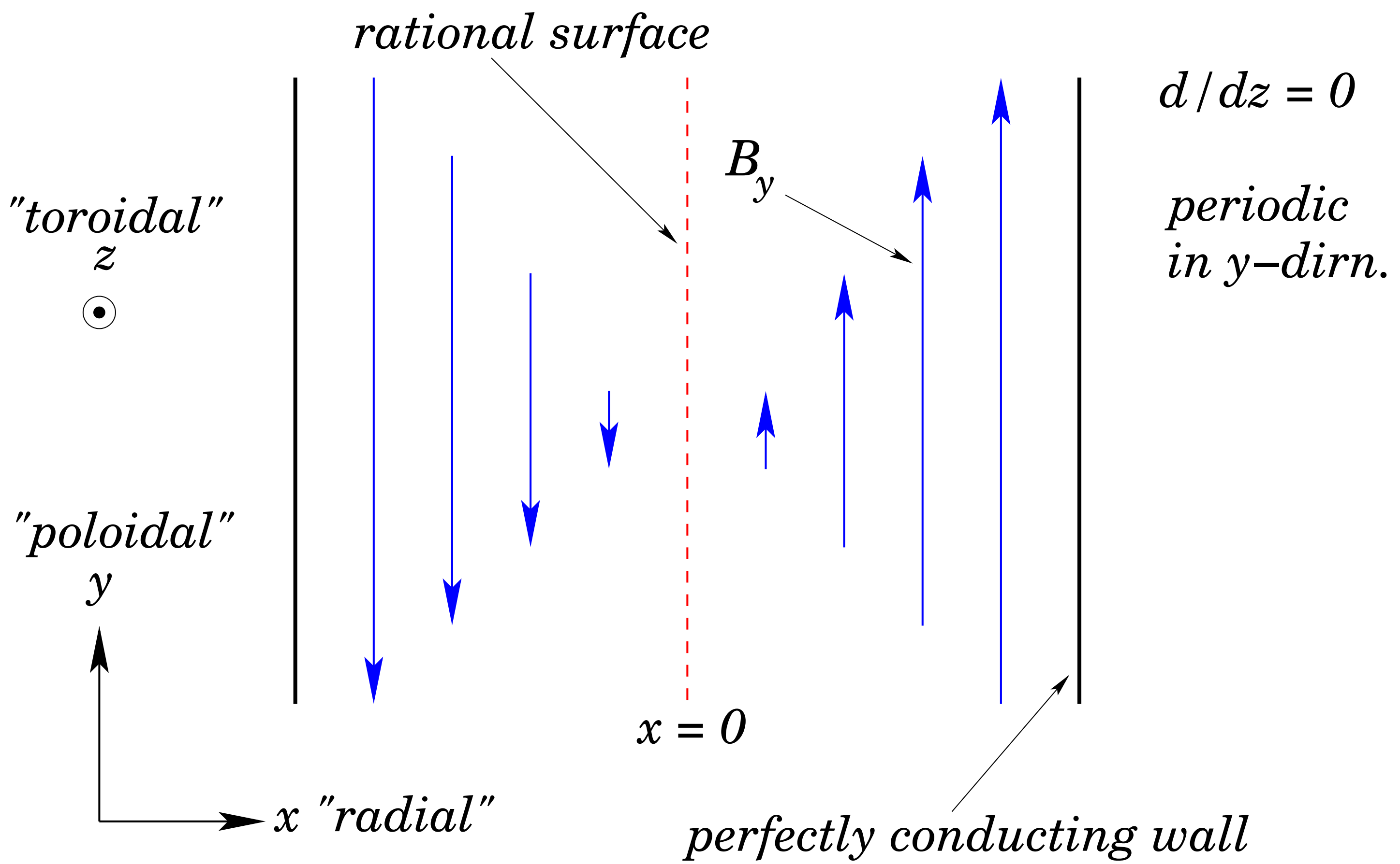


- Centered on *rational flux-surfaces* which satisfy $\vec{k} \cdot \vec{B} = 0$, where $\vec{k}$ is wave-number, and $\vec{B}$ is equilibrium magnetic field.
- Effectively "short-circuit" confinement by allowing heat to transit island region by rapidly flowing along field-lines, rather than slowly diffusing across flux-surfaces.

Introduction: Physics of Tearing Modes

- Tearing mode stability affected by great many factors:
  - MHD effects - current/pressure gradients across whole plasma.
  - Drift effects - diamagnetic flows, gyroviscosity, curvature drifts.
  - Neoclassical effects - poloidal flow-damping, bootstrap current.
  - Transport effects - anisotropic heat/particle transport.
  - Orbit effects - ion banana widths comparable to island width.
- This lecture will concentrate on fundamentals of MHD island theory.

Slab Approximation



MHD effects: Slab Model

- Cartesian coordinates: (x, y, z). Let $\partial/\partial z \equiv 0$.
- All lengths/magnetic field-strengths normalized to convenient length-scale/scale field-strength. All times normalized to associated Alfvén time.
- Harris pinch equilibrium: $B_y = \tanh(x/b)$.
- Perfect wall boundary conditions at $x = \pm a$.
- Wave-number of tearing instability: $\vec{k} = (0, k, 0)$, so $\vec{k} \cdot \vec{B} = 0$ at $x = 0$. Hence, magnetic island centered at $x = 0$.

MHD effects: Model MHD equations

- Let $\vec{B} = \nabla\psi \times \vec{z}$ and $\vec{V} = \nabla\phi \times \vec{z}$, where $\vec{V}$ is MHD fluid velocity.
- MHD equations:

$$\frac{\partial \psi}{\partial t} = [\phi, \psi] + \eta\, J,$$
$$\frac{\partial U}{\partial t} = [\phi, U] + [J, \psi] + \mu\, \nabla^2 U,$$

where $J = \nabla^2\psi$, $U = \nabla^2\phi$, and $[A, B] = A_x\, B_y - A_y\, B_x$. Here, η is resistivity, and μ is viscosity. In normalized units: $\eta, \mu \ll 1$.

MHD effects: Outer Region

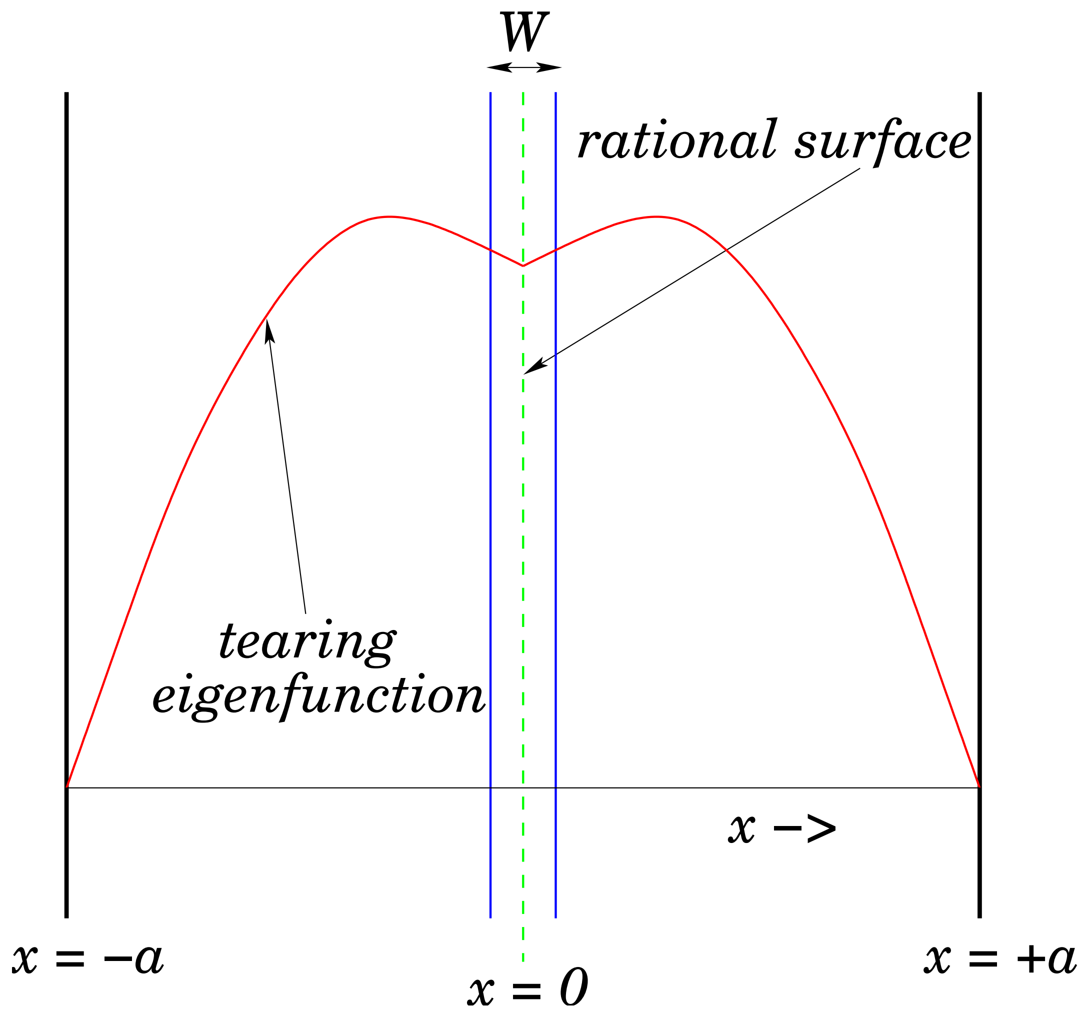
- In "outer region", which comprises most of plasma, non-ideal effects (η and μ), and plasma inertia ($\partial/\partial t$ and $\vec{V}\cdot\nabla$), negligible.
- Vorticity equation reduces to

$$[J, \psi] \simeq 0.$$

- When linearized, obtain

$$\left(\frac{d^2}{dx^2} - k^2\right)\psi^{(1)} - \left(\frac{d^2 B_y^{(0)}/dx^2}{B_y^{(0)}}\right)\psi^{(1)} = 0.$$

- Equation is *singular* at rational surface, $x = 0$, where $B_y^{(0)} = 0$.


W
rational surface
tearing
eigenfunction
x ->
x = −a
x = 0
x = +a

MHD effects: Tearing Stability Index

- Find tearing eigenfunction, $\psi^{(1)}(x)$, which is continuous, has tearing parity $[\psi^{(1)}(-x) = \psi^{(1)}(x)]$, and satisfies boundary condition $\psi^{(1)}(a) = 0$ at conducting wall.
- In general, eigenfunction has *gradient discontinuity* across rational surface (at $x = 0$). Allowed because tearing mode equation singular at rational surface.
- Tearing stability index:

$$\Delta' = \left[\frac{d \ln \psi^{(1)}}{dx}\right]_{0-}^{0+}.$$

- According to conventional MHD theory,[a] tearing mode is unstable if $\Delta' > 0$.

[a] H.P. Furth, J. Killeen, and M.N. Rosenbluth, Phys. Fluids **6**, 459 (1963).

MHD effects: Inner Region

- Inner region centered on $x = 0$. Of extent, $W \ll 1$, where W is magnetic island width (in x).
- In inner region, non-ideal effects, non-linear effects, and plasma inertia can all be important.
- Inner solution must be asymptotically matched to outer solution already obtained.

MHD effects: Constant-ψ Approximation

- Magnetic flux generally does not vary significantly in x over inner region:

$$|\psi(W, y) - \psi(0, y)| \ll |\psi(0, y)|.$$

- Constant-ψ approximation: treat ψ as constant in x over inner region.

- Approximation valid provided

$$|\Delta'|\, W \ll 1,$$

which is easily satisfied for conventional tearing modes.

MHD effects: Constant-ψ Magnetic Island

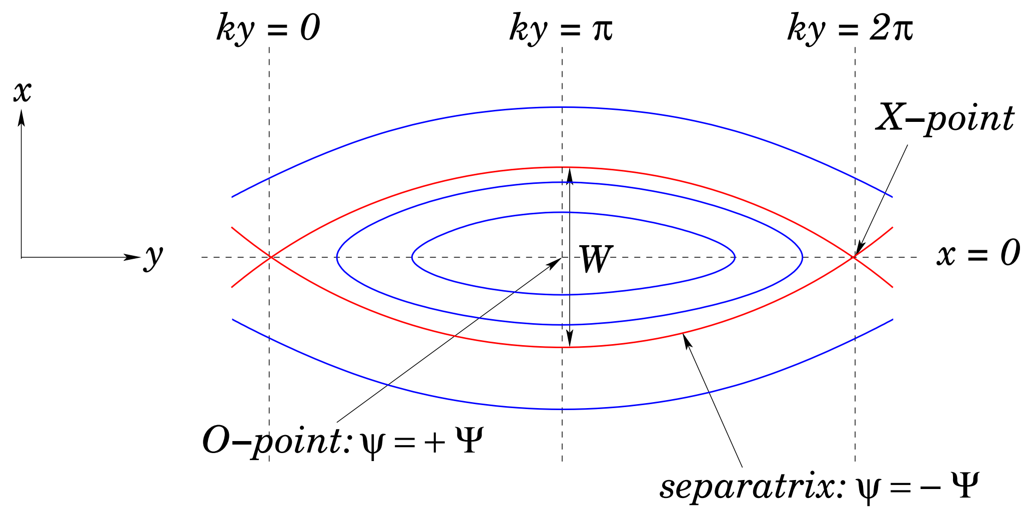
- In vicinity of rational surface:

$$\psi(x, y, t) \simeq -\frac{1}{2}\, x^2 + \Psi(t)\, \cos\theta,$$

where Ψ is "reconnected flux, and $\theta = k\, y$.

- Full island width

$$W = 4\,\sqrt{\Psi}.$$



MHD effects: Flux-Surface Average Operator

- Flux-surface average operator is annihilator of Poisson bracket $[A, \psi] \equiv \vec{B} \cdot \nabla A \equiv k\, x\, (\partial A/\partial\theta)_\psi$ for any A: *i.e.*,

$$\langle [A, \psi] \rangle = 0.$$

- Outside separatrix:

$$\langle f(\psi, \theta) \rangle = \oint \frac{f(\psi, \theta)}{|x|} \, \frac{d\theta}{2\pi}.$$

- Inside separatrix:

$$\langle f(s, \psi, \theta) \rangle = \int_{-\theta_0}^{\theta_0} \frac{f(s, \psi, \theta) + f(-s, \psi, \theta)}{2\,|x|} \, \frac{d\theta}{2\pi},$$

where $s = \mathrm{sgn}(x)$, and $x(s, \psi, \theta_0) = 0$.

MHD effects: MHD Flow -I

- Move to island frame. Look for steady-state solution: $\partial/\partial t = 0$.[a]
- Ohm's law:

$$0 \simeq [\phi, \psi] + \eta\, J.$$

- Since $\eta \ll 1$, first term potentially much larger than second.
- To lowest order:

$$[\phi, \psi] \simeq 0.$$

- Follows that

$$\phi = \phi(\psi);$$

i.e., MHD flow constrained to be around flux-surfaces.

[a] F.L. Waelbroeck, and R. Fitzpatrick, Phys. Rev. Lett. **78**, 1703 (1997).

MHD effects: MHD Flow - II

- Let

$$M(\psi) = \frac{d\phi}{d\psi}.$$

- Easily shown that

$$V_y = x\, M.$$

- $M(\psi)$ is *odd* function of x. Hence,

$$M = 0$$

inside separatrix: *i.e.*, no flow inside separatrix in island frame. Plasma trapped within separatrix.

MHD effects: MHD Flow - III

- Vorticity equation:

$$0 \simeq -M\,[U,\psi] + [J,\psi] + \mu\,\nabla^4\phi.$$

- Flux-surface average, recalling that $\langle [A,\psi] \rangle = 0$:

$$\langle \nabla^4\phi \rangle \simeq 0.$$

- Solution outside separatrix:

$$M(\psi) = \mathrm{sgn}(x)\,M_0\,\int_{-\Psi}^{\psi} d\psi/\langle x^4 \rangle \bigg/ \int_{-\Psi}^{-\infty} d\psi/\langle x^4 \rangle.$$

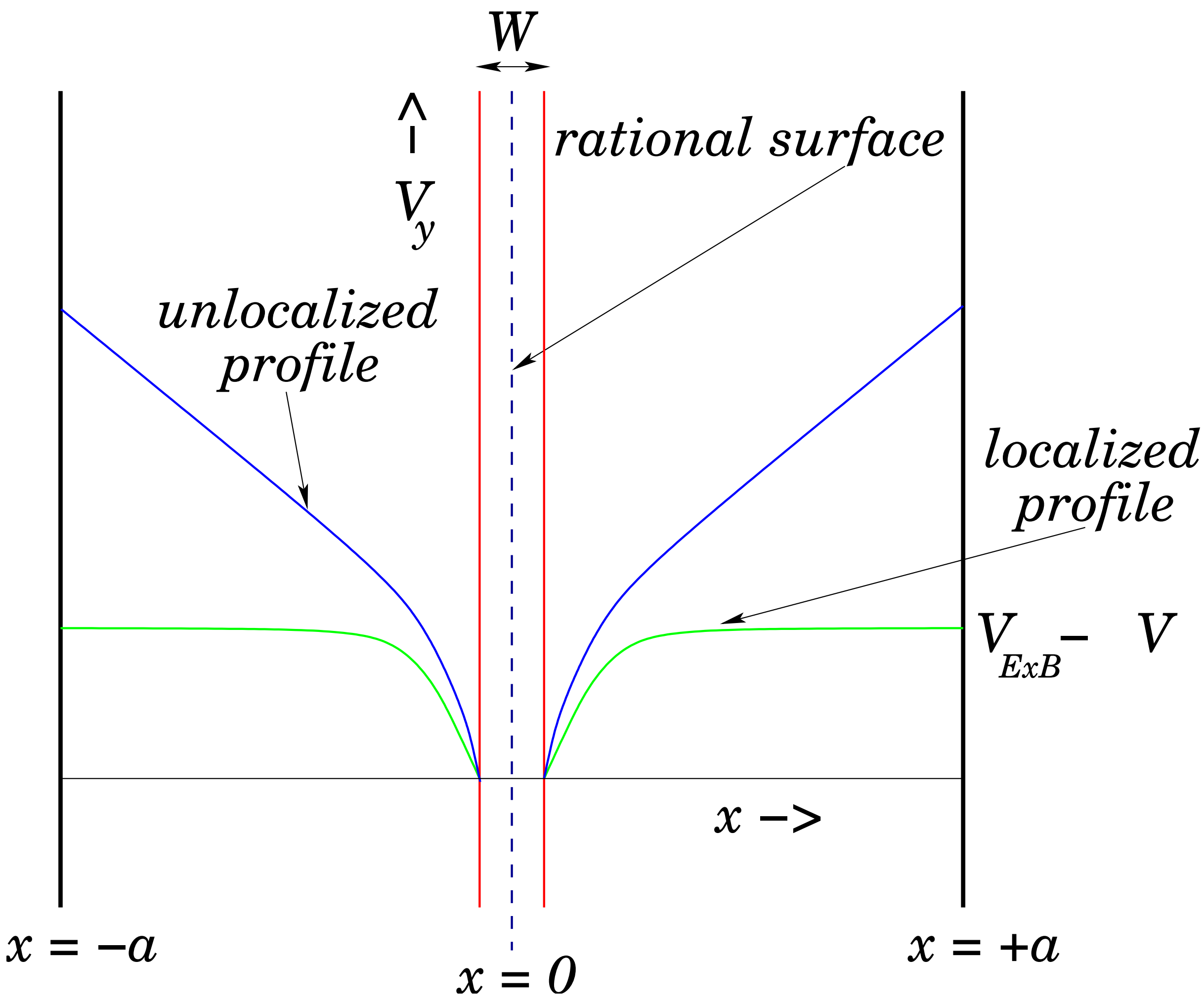
MHD effects: MHD Flow - IV

- Note

$$V_y = x\,M \to |x|\,M_0$$

  as $|x|/W \to \infty$.

- V-shaped velocity profile which extends over whole plasma.
- Expect *isolated* magnetic island to have *localized* velocity profile. Suggests that $M_0 = 0$ for isolated island.
- Hence, island propagates at local $\vec{E} \times \vec{B}$ velocity.


W
^
|
V_y
rational surface
unlocalized
profile
localized
profile
$V_{ExB} - V$
x ->
x = −a
x = 0
x = +a

MHD effects: MHD Flow - V

- Also expect isolated island to be subject to *zero EM torque*.
- Zero torque condition:

$$\int_{+\Psi}^{-\infty} \langle J\,\sin\theta\rangle\,d\psi \equiv \frac{1}{k\,\Psi}\int_{+\Psi}^{-\infty}\langle x\,[J,\psi]\rangle\,d\psi = 0.$$

- Follows from vorticity equation that

$$\lim x/w \to \infty\left[\mathrm{sgn}(x)\,x^2\,\frac{d}{dx}\left(\frac{1}{x}\frac{d\,(x\,M)}{dx}\right)\right] = -M_0 = 0.$$

- Hence, localized velocity profiles ($M_0 = 0$) automatically lead to zero EM torque, and *vice versa*.

MHD effects: Rutherford Equation - I

- Asymptotic matching between inner and outer regions yields:

$$\Delta' \Psi = -4 \int_{+\Psi}^{-\infty} \langle J \cos\theta \rangle \, d\psi.$$

- In absence of flow, in island frame, vorticity equation reduces to

$$[J, \psi] \simeq 0.$$

- Hence,

$$J = J(\psi).$$

MHD effects: Rutherford Equation - II

- Ohm's law:

$$\frac{d\Psi}{dt}\cos\theta \simeq [\phi,\psi] + \eta\, J(\psi).$$

- Flux surface average:

$$\frac{d\Psi}{dt}\langle\cos\theta\rangle \simeq \eta\, J(\psi)\,\langle 1\rangle.$$

- Hence,

$$\Delta'\,\Psi \simeq -\frac{4}{\eta}\frac{d\Psi}{dt}\int_{+\Psi}^{-\infty}\frac{\langle\cos\theta\rangle^2}{\langle 1\rangle}\,d\psi.$$

MHD effects: Rutherford Equation - III

- Evaluating flux-surface integral, obtain Rutherford equation:[a]

$$\frac{0.823}{\eta}\frac{dW}{dt} \simeq \Delta'.$$

- According to Rutherford equation, island grows algebraically on resistive time-scale.
- Rutherford equation does not predict island saturation.

[a] P.H. Rutherford, Phys. Fluids **16**, 1903 (1973).

MHD effects: Rutherford Equation - IV

- Higher order asymptotic matching between inner and outer regions yields:[a]

$$\frac{0.823}{\eta}\frac{dW}{dt} \simeq \Delta' - 0.41\left(-\frac{d^4B_y^{(0)}/dx^4}{d^2B_y^{(0)}/dx^2}\right)_{x=0} W.$$

- Hence, saturated $(d/dt = 0)$ island width is

$$W_0 = \frac{\Delta'}{0.41}\left(-\frac{d^2B_y^{(0)}/dx^2}{d^4B_y^{(0)}/dx^4}\right)_{x=0}.$$

[a] F. Militello, and F. Porcelli, Phys. Plasmas **11**, L13 (2004); D.F. Escande, and M. Ottaviani, Physics Lett. A **323**, 278 (2004).

MHD effects: Summary

- Tearing mode unstable if $\Delta' > 0$.
- Island propagates at local $\vec{E} \times \vec{B}$ velocity at rational surface.
- Island grows algebraically on resistive time-scale.
- Saturated island width:

$$W_0 = \frac{\Delta'}{0.41} \left(- \frac{d^2 B_y^{(0)}/dx^2}{d^4 B_y^{(0)}/dx^4} \right)_{x=0}.$$