Introduction to Magnetic Island Theory [a]

Richard Fitzpatrick

Institute for Fusion Studies

University of Texas at Austin

Austin, TX, USA

[a] Lectures based on work of R. Fitzpatrick, F.L. Waelbroeck, and F. Militello.

Lecture 3

Supersonic Islands:[a] Drift-MHD Equations

- Steady-state drift-MHD equations (with $\tau = 0$, since ion diamagnetic effects largely irrelevant to supersonic islands):

$$\begin{aligned}
\psi &= -x^2/2 + \Psi \cos\theta, \qquad U = \nabla^2\phi, \\
0 &= [\phi - n, \psi] + \eta\, J, \\
0 &= [\phi, U] + [J, \psi] + \mu_i\, \nabla^4\phi, \\
0 &= [\phi, n] + [V_z + J, \psi] + D\, \nabla^2 n, \\
0 &= [\phi, V_z] + \alpha\, [n, \psi] + \mu_i \nabla^2 V_z.
\end{aligned}$$

[a] R. Fitzpatrick, and F.L. Waelbroeck, preprint (2007).

Supersonic Islands: Zero-α Solution

- By definition, highlighted term small for supersonic islands.
- If term completely neglected, obtain trivial solution:

$$\phi = n = -x, \qquad U = V_z = J = 0.$$

- Island propagates with *electron fluid*.
- Island does not perturb ion fluid, so *zero polarization current*.

Supersonic Islands: Small-α Solution

- Assume that highlighted term small, but not negligible. Perturb about zero-α solution.
- So

$$\phi = -x + \delta\,\phi, \qquad n = -x + \delta n,$$

where $\delta\phi$, δn, U, V_z, J all $O(\alpha) \ll 1$.

Supersonic Islands: Analysis - I

- Lowest order solution:

$$\begin{aligned}\delta n &= \delta\phi + H(\psi), \\ J &= -\widetilde{G} + (\alpha/2)\,\widetilde{x^2}, \\ V_z &= -\alpha\,(W/4)^2\,\cos\theta,\end{aligned}$$

where $\widetilde{A} \equiv A - \langle A\rangle/\langle 1\rangle$.

- Here, $G = -x\,H'$. Now, $G = 0$ inside separatrix, but outside separatrix

$$G = |x|\left(\frac{\langle x\,v\rangle + \alpha\,(W/4)^4}{\langle x^2\rangle}\right),$$

where $v = -\delta\phi_x$.

Supersonic Islands: Analysis - II

- Perturbed velocity v satisfies

$$v_{xx} = (D/\mu)\,(\overline{v} - \overline{G}) - (G - \overline{G}) - \alpha\,(W/4)^2\,\cos\theta,$$

where $\overline{\cdots}$ denotes a θ-average at constant x.

- Boundary conditions: $v_x = 0$ at $x = 0$, and

$$v \to v_i + v_i'\,|x| - (\alpha/2)\,(W/4)^2\,x^2\,\cos\theta$$

as $|x| \to \infty$.

- Above equation highly nonlinear, but can be solved via iteration.

Supersonic Islands: Need for Intermediate Layer

- Inner region island solution does not satsify $J \to 0$ as $|x| \to \infty$: *i.e.*, it does not asymptote to ideal-MHD solution in outer region.
- Require *intermediate layer* between island and outer region to allow proper matching.
- Intermediate layer much wider than island, so governed by linear physics.

Supersonic Islands: Intermediate Layer - I

- Write

$$\phi(x,\theta) = -x + \overline{\delta\phi}(x) + \breve{\phi}(x)\,e^{i\,\theta}.$$

- Neglect all transport terms except ion viscosity.
- Linearized drift-MHD equations yield

$$\begin{aligned}\breve{\phi}_{xx} &- \bar{v}_{xx}\,\breve{\phi} - \left(\bar{v} - \frac{\alpha\,x^2}{1 - i\,\mu_i\,\alpha\,x^2}\right)\breve{\phi} \\ &= -\left(\bar{v} - \frac{\alpha\,x^2}{1 - i\,\mu_i\,\alpha\,x^2}\right)\frac{(W/4)^2}{x},\end{aligned}$$

where $\bar{v} = -\overline{\delta\phi}_x$.

Supersonic Islands: Intermediate Layer - II

- Mean velocity profile determined by *quasi-linear force balance*:

$$\bar{v}_{xx} = \frac{1}{2}\,\frac{\alpha^2\,x^2}{1+(\mu_i\,\alpha\,x^2)^2}\,|(W/4)^2 - x\,\breve{\phi}|^2.$$

- Perturbed current:

$$\breve{J} = (\breve{\phi}_{xx} - \bar{v}_{xx}\,\breve{\phi})/x.$$

Supersonic Islands: Intermediate Layer - III

- Boundary conditions as $x \to 0$:

$$\begin{aligned} \breve{\phi} &\to 0, \\ \bar{v} &\to v_i + v_i' \, |x|. \end{aligned}$$

- Boundary conditions as $|x| \to \infty$:

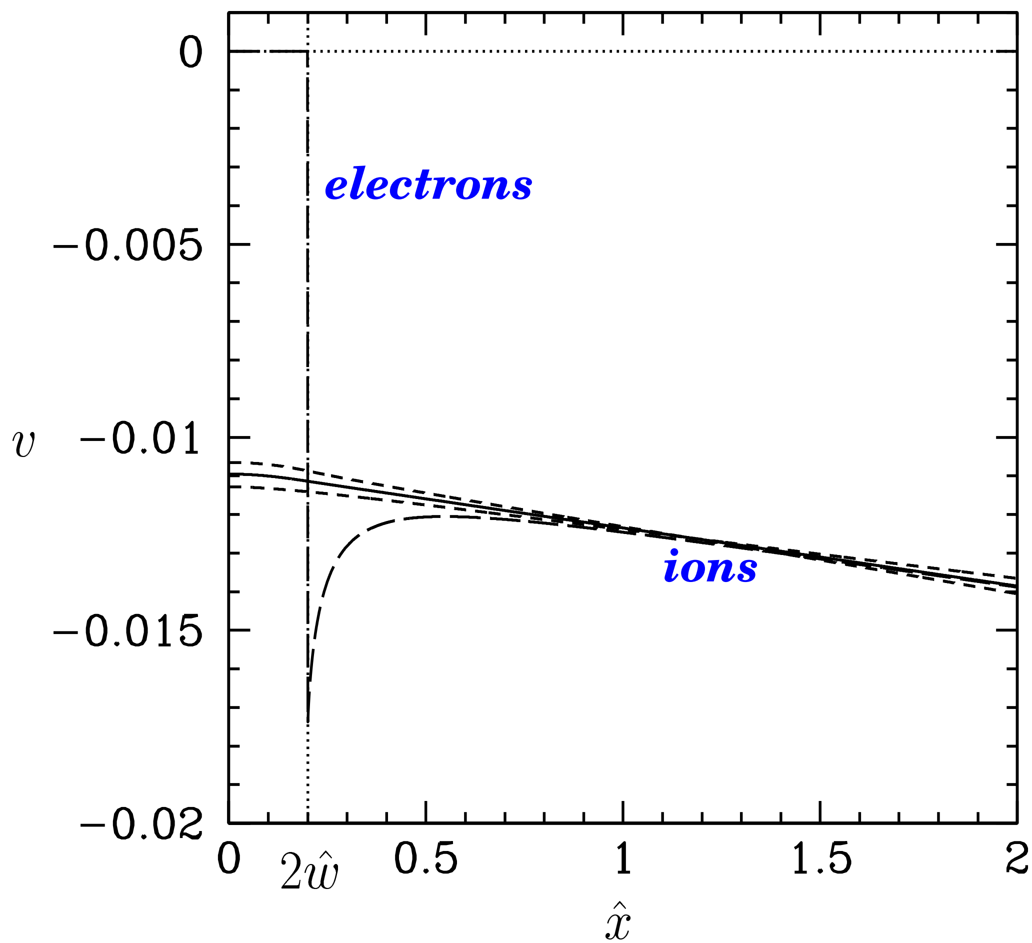
$$\begin{aligned} \breve{\phi} &\to (W/4)^2/x, \\ \bar{v} &\to v_\infty + v_\infty' \, |x|. \end{aligned}$$

- Large-$|x|$ boundary conditions ensure that $\breve{J} \to 0$. So solution matches to ideal-MHD solution.

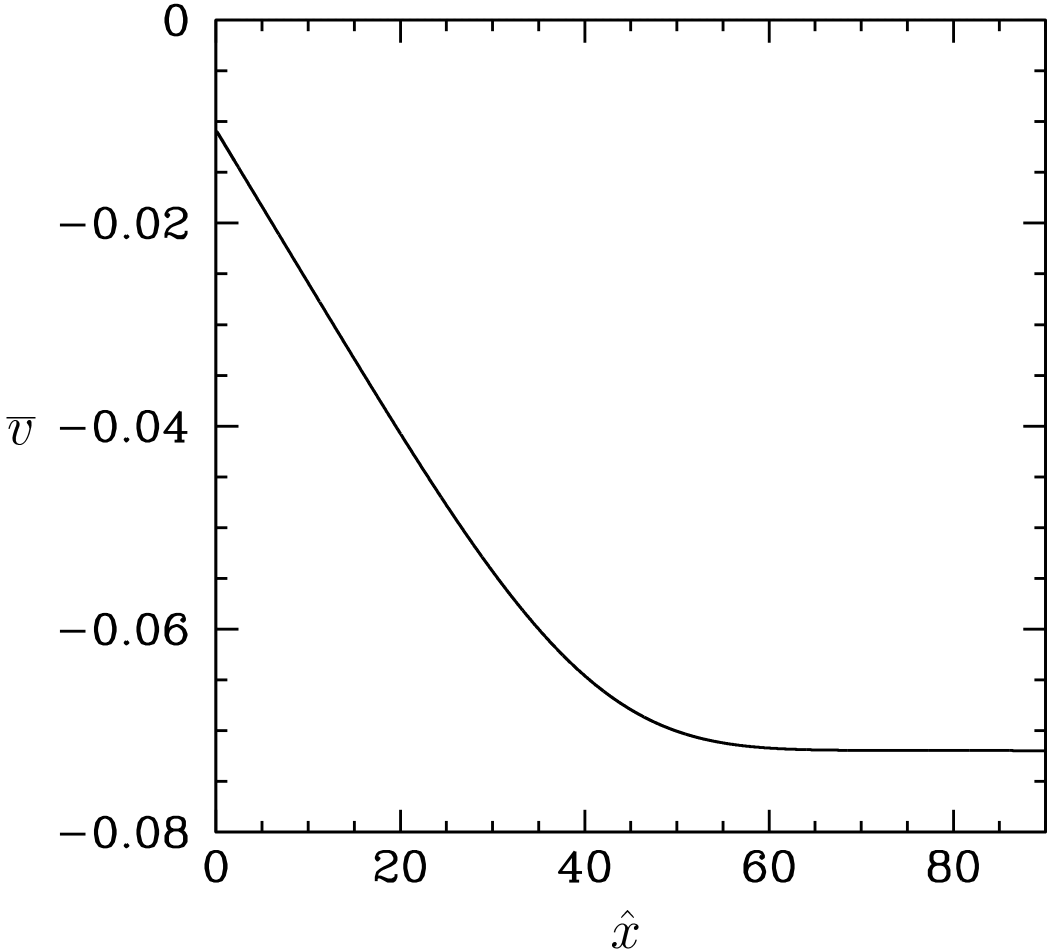
Supersonic Islands: Physics of Intermediate Layer

- Island launches *drift-acoustic waves* into intermediate layer.
- Waves are *absorbed* in layer (due to ion viscosity).
- Waves carry *momentum*.
- Momentum exchange between island and intermediate layer ensures that velocity gradient, v_i', at inner boundary of layer not same as gradient, v_∞', at outer boundary.
- For isolated island solution, require $v_\infty' = 0$. This boundary condition *uniquely specifies* solution for given values of α, μ_i, D, etc.

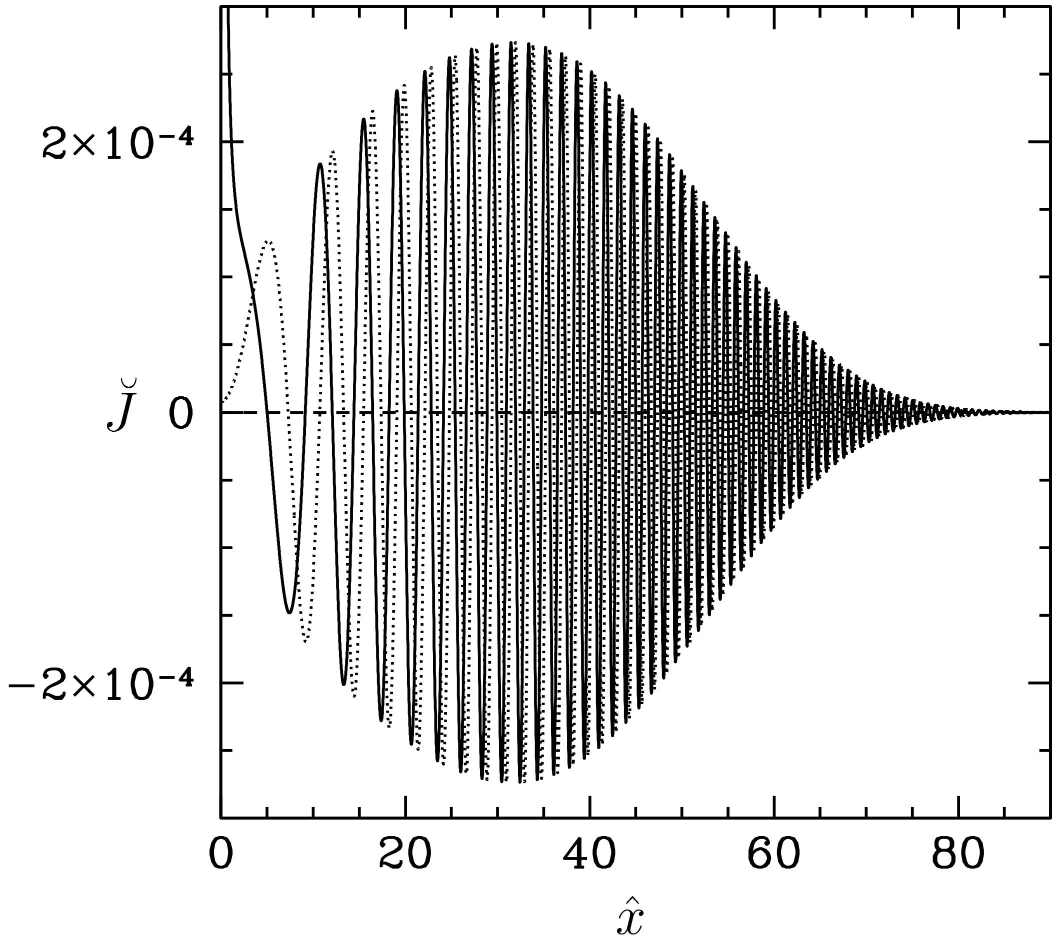
Supersonic Islands: Velocity in Island Region



Supersonic Islands: Velocity in Intermediate Layer



Supersonic Islands: Current in Intermediate Layer



Supersonic Islands: Island Propagation

- Island propagation velocity:

$$V = V_e - 0.27\,(W/4)^3\,\alpha^{3/4}\,D^{-1} - 0.24\,(W/4)^4\,\alpha^{1/3}\,\mu^{-4/3}.$$

- Island phase velocity close to unperturbed electron fluid velocity, but dragged slightly in ion direction due to sound-wave effects.

Supersonic Islands: Polarization Term

- Rutherford Equation:

$$\frac{0.823}{\eta}\,\frac{dW}{dt} = \Delta' - 1.5\,\beta\,\alpha^{-1/4} - 0.38\,\beta\,\alpha^{-1/4}\,(W/4)^2\,D^{-1}.$$

- Sound-wave effects ensure ion fluid slightly perturbed by island, generating polarization term in Rutherford equation. Term is *stabilizing*.

Supersonic Islands: Maximum Island Width

- Supersonic branch of solutions ceases to exist beyond *maximum island width*:

$$W_{max} = 0.36\,\alpha^{-1/12}\,D^{1/3}.$$

- Hypothesized that island bifurcates to subsonic solution branch when $W > W_{max}$. This type of behavior has been observed in computer simulations.[a]

[a]M. Ottaviani, F. Porcelli, and D. Grasso, Phys. Rev. Lett. **93**, 075001 (2004).

Supersonic Islands: Summary

- Results limited to small islands: *i.e.*, small enough that sound waves cannot flatten density profile.
- Islands phase velocities close to unperturbed electron fluid velocity, but dragged slightly in ion direction by sound wave effects.
- Islands radiate drift-acoustic waves.
- Momentum carried by drift-acoustic waves gives rise to strong velocity shear in region surrounding islands.
- Polarization term in Rutherford island equation is stabilizing.
- Supersonic branch ceases to exist above critical island width.

Further Work - I

- Need to develop theory of subsonic branch of solutions in limit
$$W \ll L_s/L_n.$$
- Such theory will determine island solution to which supersonic branch bifurcates when maximum island width exceeded.
- Theory difficult because drift-acoustic resonance located in nonlinear region.
- Does subsonic solution branch cease to exist below some critical island width?

Further Work - II

- Width of trapped ion orbit of order

$$\rho_\theta = (B_z/B_y)\,\rho.$$

- Hence likely that trapped ion orbit width comparable with island width.

- Need to incorporate this effect into analysis. [a]

[a] A. Bergmann, E. Poli, and A.G. Peeters, Phys. Plasmas **12**, 072501 (2005).

Further Work - III

- Magnetic *field-line curvature* in toroidal confinement devices gives rise to particle drifts which are three-dimensional in nature, and cannot be captured in two-dimensional slab model.
- Need to develop *three-dimensional* island theory which takes curvature drifts into account. [a]

[a] M. Kotschenreuther, R.D. Hazeltine, and P.J. Morrison, Phys. Fluids **28**, 294 (1985).

Further Work - IV

- Perpendicular transport which determines island profiles actually due to *drift-wave turbulence*.
- Radial extent of drift-wave eddies of order ρ. Hence, eddies can be comparable in width to island.
- Need to develop island theory in which island immersed in bath of drift-wave turbulence. Turbulence effects island by modifying island profiles. Island profiles effect drift-wave stability, and hence turbulence levels. Theory must *self-consistently* determine effect of turbulence on island, and effect of island on turbulence. [a]

[a] F. Militello, F.L. Waelbroeck, R. Fitzpatrick, and W. Horton, preprint (2007).