

# Introduction to Magnetic Island Theory<sup>a</sup>

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<sup>a</sup>Lectures based on work of R. Fitzpatrick, F.L. Waelbroeck, and F. Militello.

# Lecture 3

## Supersonic Islands:<sup>a</sup> Drift-MHD Equations

- Steady-state drift-MHD equations (with  $\tau = 0$ , since ion diamagnetic effects largely irrelevant to supersonic islands):

$$\psi = -x^2/2 + \Psi \cos \theta, \quad \mathcal{U} = \nabla^2 \phi,$$

$$0 = [\phi - n, \psi] + \eta J,$$

$$0 = [\phi, \mathcal{U}] + [J, \psi] + \mu_i \nabla^4 \phi,$$

$$0 = [\phi, n] + [V_z + J, \psi] + D \nabla^2 n,$$

$$0 = [\phi, V_z] + \alpha [n, \psi] + \mu_i \nabla^2 V_z.$$

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<sup>a</sup>R. Fitzpatrick, and F.L. Waelbroeck, preprint (2007).

## Supersonic Islands: Zero- $\alpha$ Solution

- By definition, highlighted term small for supersonic islands.
- If term completely neglected, obtain trivial solution:

$$\phi = n = -x, \quad U = V_z = J = 0.$$

- Island propagates with *electron fluid*.
- Island does not perturb ion fluid, so *zero polarization current*.

## Supersonic Islands: Small- $\alpha$ Solution

- Assume that highlighted term small, but not negligible. Perturb about zero- $\alpha$  solution.
- So

$$\phi = -x + \delta \phi, \quad n = -x + \delta n,$$

where  $\delta \phi, \delta n, U, V_z, J$  all  $O(\alpha) \ll 1$ .

## Supersonic Islands: Analysis - I

- Lowest order solution:

$$\delta n = \delta\phi + H(\psi),$$

$$J = -\tilde{G} + (\alpha/2) \tilde{x}^2,$$

$$V_z = -\alpha (W/4)^2 \cos \theta,$$

where  $\tilde{A} \equiv A - \langle A \rangle / \langle 1 \rangle$ .

- Here,  $G = -x H'$ . Now,  $G = 0$  inside separatrix, but outside separatrix

$$G = |x| \left( \frac{\langle x v \rangle + \alpha (W/4)^4}{\langle x^2 \rangle} \right),$$

where  $v = -\delta\phi_x$ .

## Supersonic Islands: Analysis - II

- Perturbed velocity  $v$  satisfies

$$v_{xx} = (D/\mu) (\bar{v} - \bar{G}) - (G - \bar{G}) - \alpha (W/4)^2 \cos \theta,$$

where  $\bar{\cdot}$  denotes a  $\theta$ -average at constant  $x$ .

- Boundary conditions:  $v_x = 0$  at  $x = 0$ , and

$$v \rightarrow v_i + v'_i |x| - (\alpha/2) (W/4)^2 x^2 \cos \theta$$

as  $|x| \rightarrow \infty$ .

- Above equation highly nonlinear, but can be solved via iteration.

## Supersonic Islands: Need for Intermediate Layer

- Inner region island solution does not satisfy  $J \rightarrow 0$  as  $|x| \rightarrow \infty$ : *i.e.*, it does not asymptote to ideal-MHD solution in outer region.
- Require *intermediate layer* between island and outer region to allow proper matching.
- Intermediate layer much wider than island, so governed by *linear* physics.

## Supersonic Islands: Intermediate Layer - I

- Write

$$\phi(x, \theta) = -x + \overline{\delta\phi}(x) + \check{\phi}(x) e^{i\theta}.$$

- Neglect all transport terms except ion viscosity.
- Linearized drift-MHD equations yield

$$\begin{aligned} \check{\phi}_{xx} &= \bar{v}_{xx} \check{\phi} - \left( \bar{v} - \frac{\alpha x^2}{1 - i\mu_i \alpha x^2} \right) \check{\phi} \\ &= - \left( \bar{v} - \frac{\alpha x^2}{1 - i\mu_i \alpha x^2} \right) \frac{(W/4)^2}{x}, \end{aligned}$$

where  $\bar{v} = -\overline{\delta\phi}_x$ .

## Supersonic Islands: Intermediate Layer - II

- Mean velocity profile determined by *quasi-linear force balance*:

$$\bar{v}_{xx} = \frac{1}{2} \frac{\alpha^2 x^2}{1 + (\mu_i \alpha x^2)^2} |(\mathcal{W}/4)^2 - x \check{\phi}|^2.$$

- Perturbed current:

$$\check{J} = (\check{\phi}_{xx} - \bar{v}_{xx} \check{\phi})/x.$$

## Supersonic Islands: Intermediate Layer - III

- Boundary conditions as  $x \rightarrow 0$ :

$$\check{\phi} \rightarrow 0,$$

$$\bar{v} \rightarrow v_i + v'_i |x|.$$

- Boundary conditions as  $|x| \rightarrow \infty$ :

$$\check{\phi} \rightarrow (W/4)^2 / x,$$

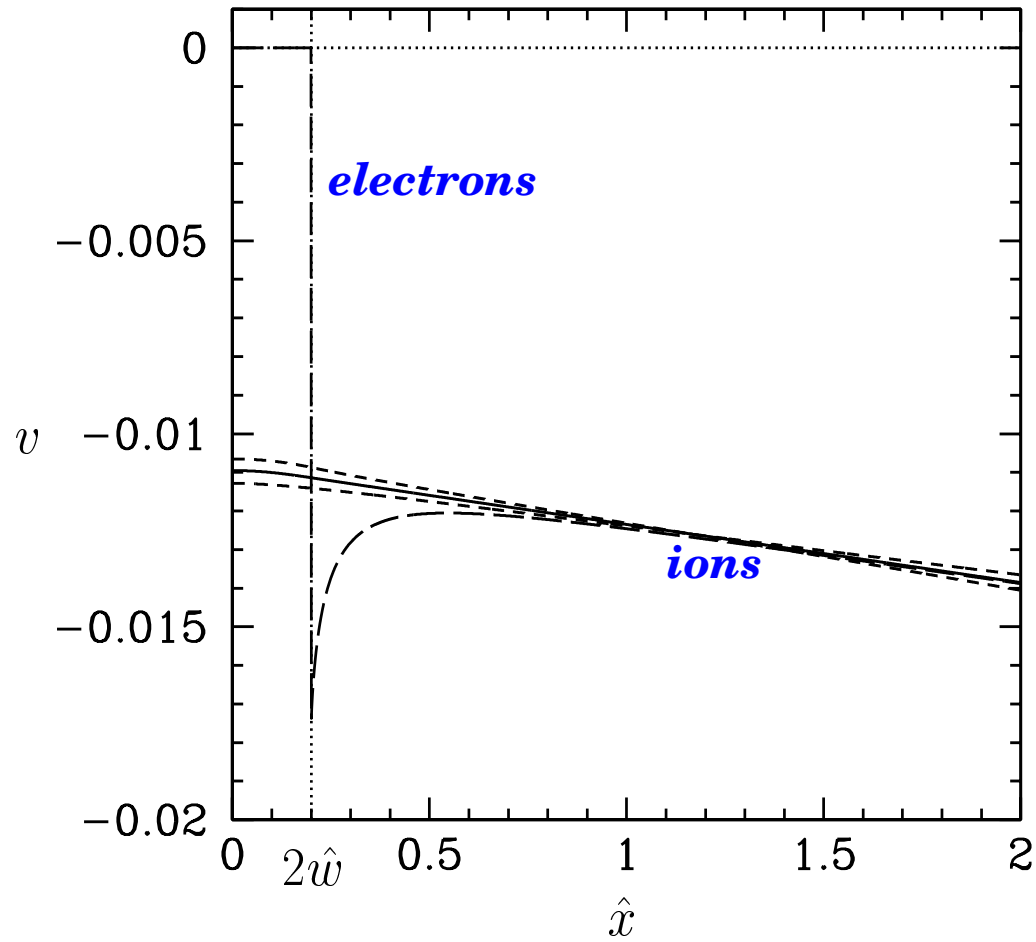
$$\bar{v} \rightarrow v_\infty + v'_\infty |x|.$$

- Large- $|x|$  boundary conditions ensure that  $\check{j} \rightarrow 0$ . So solution matches to ideal-MHD solution.

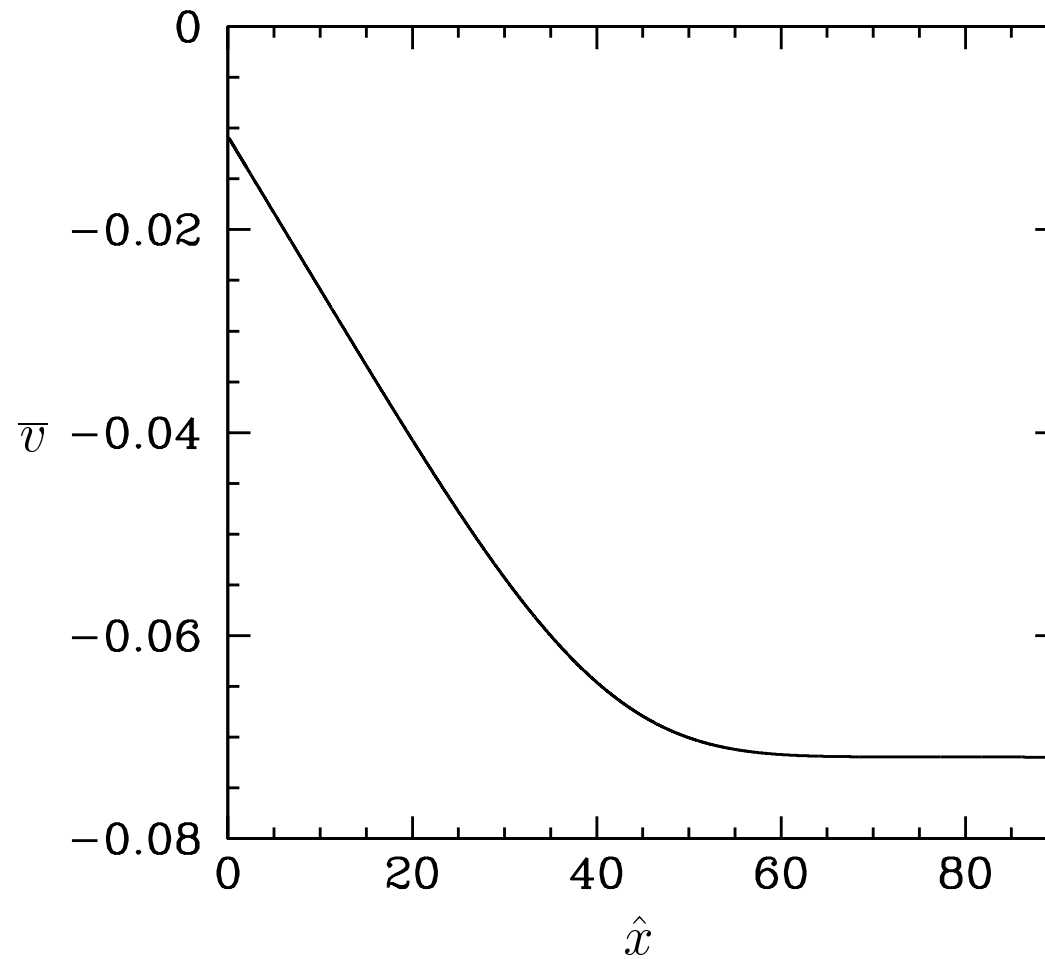
## Supersonic Islands: Physics of Intermediate Layer

- Island launches *drift-acoustic waves* into intermediate layer.
- Waves are *absorbed* in layer (due to ion viscosity).
- Waves carry *momentum*.
- Momentum exchange between island and intermediate layer ensures that velocity gradient,  $v'_i$ , at inner boundary of layer not same as gradient,  $v'_\infty$ , at outer boundary.
- For isolated island solution, require  $v'_\infty = 0$ . This boundary condition *uniquely specifies* solution for given values of  $\alpha$ ,  $\mu_i$ ,  $D$ , etc.

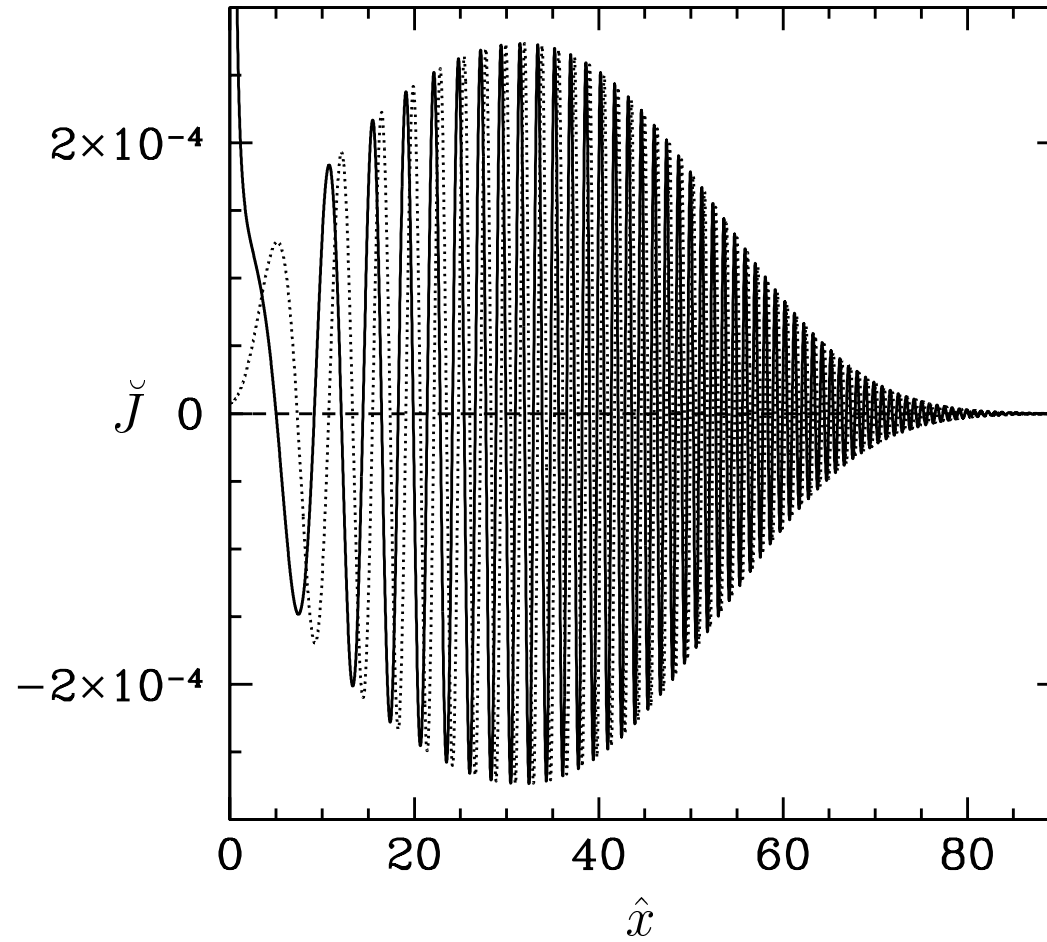
## Supersonic Islands: Velocity in Island Region



## Supersonic Islands: Velocity in Intermediate Layer



## Supersonic Islands: Current in Intermediate Layer



## Supersonic Islands: Island Propagation

- Island propagation velocity:

$$V = V_e - 0.27 (W/4)^3 \alpha^{3/4} D^{-1} - 0.24 (W/4)^4 \alpha^{1/3} \mu^{-4/3}.$$

- Island phase velocity close to unperturbed electron fluid velocity, but dragged slightly in ion direction due to sound-wave effects.

## Supersonic Islands: Polarization Term

- Rutherford Equation:

$$\frac{0.823}{\eta} \frac{dW}{dt} = \Delta' - 1.5 \beta \alpha^{-1/4} - 0.38 \beta \alpha^{-1/4} (W/4)^2 D^{-1}.$$

- Sound-wave effects ensure ion fluid slightly perturbed by island, generating polarization term in Rutherford equation. Term is *stabilizing*.

## Supersonic Islands: Maximum Island Width

- Supersonic branch of solutions ceases to exist beyond *maximum island width*:

$$W_{\max} = 0.36 \alpha^{-1/12} D^{1/3}.$$

- Hypothesized that island bifurcates to subsonic solution branch when  $W > W_{\max}$ . This type of behavior has been observed in computer simulations.<sup>a</sup>

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<sup>a</sup>M. Ottaviani, F. Porcelli, and D. Grasso, Phys. Rev. Lett. **93**, 075001 (2004).

## Supersonic Islands: Summary

- Results limited to small islands: *i.e.*, small enough that sound waves cannot flatten density profile.
- Islands phase velocities close to unperturbed electron fluid velocity, but dragged slightly in ion direction by sound wave effects.
- Islands radiate drift-acoustic waves.
- Momentum carried by drift-acoustic waves gives rise to strong velocity shear in region surrounding islands.
- Polarization term in Rutherford island equation is stabilizing.
- Supersonic branch ceases to exist above critical island width.

## Further Work - I

- Need to develop theory of subsonic branch of solutions in limit

$$W \ll L_s/L_n.$$

- Such theory will determine island solution to which supersonic branch bifurcates when maximum island width exceeded.
- Theory difficult because drift-acoustic resonance located in nonlinear region.
- Does subsonic solution branch cease to exist below some critical island width?

## Further Work - II

- Width of trapped ion orbit of order

$$\rho_{\theta} = (B_z/B_y) \rho.$$

- Hence likely that trapped ion orbit width comparable with island width.
- Need to incorporate this effect into analysis.<sup>a</sup>

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<sup>a</sup>A. Bergmann, E. Poli, and A.G. Peeters, Phys. Plasmas **12**, 072501 (2005).

## Further Work - III

- Magnetic *field-line curvature* in toroidal confinement devices gives rise to particle drifts which are three-dimensional in nature, and cannot be captured in two-dimensional slab model.
- Need to develop *three-dimensional* island theory which takes curvature drifts into account.<sup>a</sup>

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<sup>a</sup>M. Kotschenreuther, R.D. Hazeltine, and P.J. Morrison, Phys. Fluids **28**, 294 (1985).

## Further Work - IV

- Perpendicular transport which determines island profiles actually due to *drift-wave turbulence*.
- Radial extent of drift-wave eddies of order  $\rho$ . Hence, eddies can be comparable in width to island.
- Need to develop island theory in which island immersed in bath of drift-wave turbulence. Turbulence effects island by modifying island profiles. Island profiles effect drift-wave stability, and hence turbulence levels. Theory must *self-consistently* determine effect of turbulence on island, and effect of island on turbulence.<sup>a</sup>

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<sup>a</sup>F. Militello, F.L. Waelbroeck, R. Fitzpatrick, and W. Horton, preprint (2007).