Introduction to Magnetic Island Theory [a]

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[a] Lectures based on work of R. Fitzpatrick, F.L. Waelbroeck, and F. Militello.

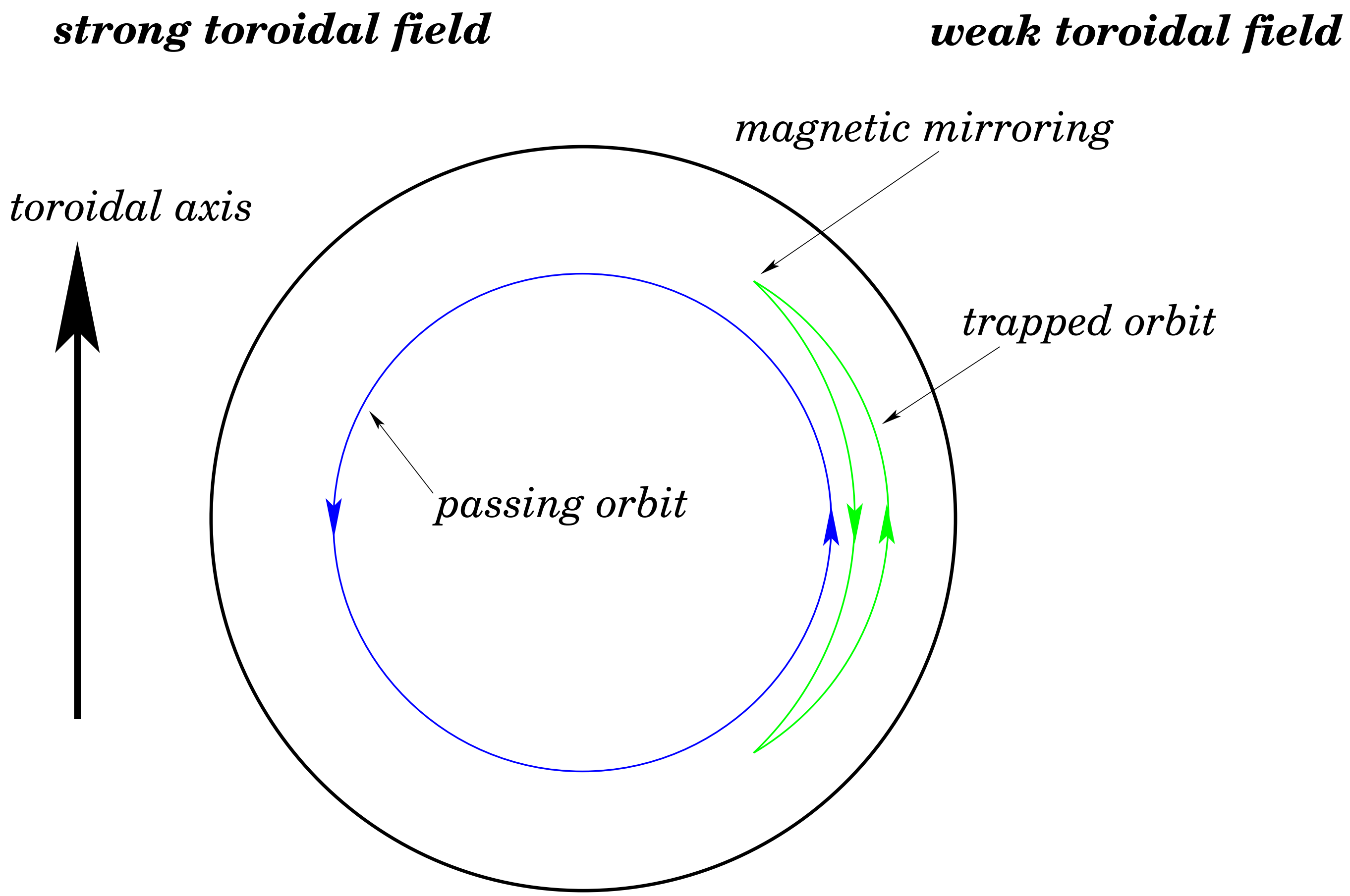
Lecture 2

Neoclassical Effects: Introduction

- So-called *neoclassical effects*[a] in magnetic confinement devices arise from combination of essential *toroidicity* of such devices, and extremely long mean-free-path of electrons and ions streaming along field-lines, due to very low collisionality of hot fusion plasmas.

[a] *The Theory of Toroidally Confined Plasmas*, 2nd Rev. Edition, R.B. White (World Scientific, 2006).

Neoclassical Effects: Trapped and Passing Particles



Neoclassical Effects: Bootstrap Current - I

- In toroidal plasma, friction between trapped and passing electrons leads to appearance of non-inductive *bootstrap current* in Ohm's law: [a]

$$\frac{d\Psi}{dt}\cos\theta \simeq [\phi, \psi] + \eta\,[J(\psi) - J_{boot}],$$

where

$$J_{boot} = -1.46\,\sqrt{\epsilon}\,B_\theta^{-1}\,\frac{\partial P}{\partial r}.$$

Here, ϵ is inverse aspect-ratio, $1.46\,\sqrt{\epsilon}$ is measure of fraction of trapped-particles, and P is plasma pressure.

[a] M.N. Rosenbluth, R.D. Hazeltine, and F.L. Hinton, Phys. Fluids **15**, 116 (1972).

Neoclassical Effects: Bootstrap Current - II

- Pressure profile often *flattened* inside island separatrix.
- Bootstrap current consequently disappears inside separatrix.
- Absence of bootstrap current inside separatrix, and continued presence outside, leads to *destabilizing* term in Rutherford island equation:[a]

$$\frac{0.823}{\eta}\frac{dW}{dt} \simeq \Delta' - 2.31\sqrt{\epsilon}\,\frac{(r\,P'/B_\theta^2)}{(W/4)}$$

[a] R. Fitzpatrick, Phys. Plasmas **2**, 825 (1995).

Neoclassical Effects: Neoclassical Tearing Modes - I

- A *neoclassical tearing mode* (NTM) is an *intrinsically stable* ($\Delta' < 0$) tearing mode destabilized by bootstrap term.
- Bootstrap term in Rutherford equation relatively large, especially at small island widths. Would expect plasma to be filled with NTMs, and confinement to be wrecked.
- This is not observed to be case. Experimental evidence for threshold island width above which NTMs grow, but below which they decay.[a]
- Suggests presence of stabilizing term in Rutherford equation which opposes destabilizing bootstrap term.

[a] O. Sauter, *et al.*, Phys. Plasmas **4**, 1654 (1997).

Neoclassical Effects: Neoclassical Tearing Modes - II

- Most likely candidate for stabilizing term in Rutherford equation, which provides NTM threshold mechanism, is well-known term due to ion polarization current.[a]
- In order to investigate this term, must graduate to two-fluid drift-MHD magnetic island theory.

[a] A.I. Smolyakov, Sov. J. Plasma Phys. **15**, 667 (1989).

Drift-MHD Theory: Introduction

- In drift-MHD approximation, analysis retains *charged particle drift velocities*, in addition to $\vec{E} \times \vec{B}$ velocity.
- Essentially two-fluid theory of plasma.
- Characteristic length-scale, ρ, is *ion Larmor radius calculated with electron temperature*.
- Characteristic velocity is diamagnetic velocity, V_*, where

$$n\,e\,\vec{V}_* \times \vec{B} = \nabla P.$$

- Normalize all lengths to ρ, and all velocities to V_*.

Drift-MHD Theory: Basic Assumptions

- Retain slab model, for sake of simplicity.
- Assume parallel electron heat transport sufficiently strong that $T_e = T_e(\psi)$.
- Assume $T_i/T_e = \tau = \mathrm{constant}$, for sake of simplicity.

Drift-MHD Theory: Basic Definitions

- Variables:
  - ψ - magnetic flux-function.
  - J - parallel current.
  - ϕ - guiding-center (*i.e.*, MHD) stream-function.
  - U - parallel ion vorticity.
  - n - electron number density (minus uniform background).
  - V_z - parallel ion velocity.
- Parameters:
  - $\alpha = (L_n/L_s)^2$, where L_n is equilibrium density gradient scale-length.
  - η - resistivity. D - (perpendicular) particle diffusivity. $\mu_{i/e}$ - (perpendicular) ion/electron viscosity.

Drift-MHD Theory: Drift-MHD Equations - I

- Steady-state drift-MHD equations:[a]

$$
\begin{aligned}
\psi &= -x^2/2 + \Psi\cos\theta, \qquad U = \nabla^2\phi, \\
0 &= [\phi - n, \psi] + \eta\, J, \\
0 &= [\phi, U] - \frac{\tau}{2}\left\{\nabla^2[\phi, n] + [U, n] + [\nabla^2 n, \phi]\right\} \\
&\quad + [J, \psi] + \mu_i \nabla^4(\phi + \tau\, n) + \mu_e\, \nabla^4(\phi - n), \\
0 &= [\phi, n] + [V_z + J, \psi] + D\,\nabla^2 n, \\
0 &= [\phi, V_z] + \alpha\,[n, \psi] + \mu_i\,\nabla^2 V_z.
\end{aligned}
$$

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[a] R.D. Hazeltine, M. Kotschenreuther, and P.J. Morrison, Phys. Fluids **28**, 2466 (1985).

Drift-MHD Theory: Drift-MHD Equations - II

- Symmetry: ψ, J, V_z even in x. ϕ, n, U odd in x.
- Boundary conditions as $|x|/W \to \infty$:
  - $n \to -(1+\tau)^{-1}\,x$.
  - $\phi \to -V\,x$.
  - $J, U, V_z \to 0$.
- Here, V is island phase-velocity in $\vec{E}\times\vec{B}$ frame.
- $V = 1$ corresponds to island propagating with electron fluid. $V = -\tau$ corresponds to island propagating with ion fluid.
- Expect

$$1 \gg \alpha \gg \eta, D, \mu_i, \mu_e.$$

Drift-MHD Theory: Electron Fluid

- Ohm's law:

$$0 = [\phi - n, \psi] + \eta\, J.$$

- Since $\eta \ll 1$, first term potentially much larger than second.
- To lowest order:

$$[\phi - n, \psi] \simeq 0.$$

- Follows that

$$n = \phi + H(\psi):$$

  i.e., electron stream-function $\phi_e = \phi - n$ is *flux-surface function*. Electron fluid flow constrained to be around flux-surfaces.

Drift-MHD Theory: Sound Waves

- Parallel flow equation:

$$0 = [\phi, V_z] + \alpha\,[n, \psi] + \mu_i\,\nabla^2 V_z.$$

- Highlighted term dominant provided

$$W \gg \alpha^{-1/2} = L_s/L_n.$$

- If this is case, then to lowest order

$$n = n(\psi),$$

  which implies $n = 0$ inside separatrix.

- So, if island sufficiently wide, *sound-waves* able to *flatten density profile* inside island separatrix.

Drift-MHD Theory: Subsonic vs. Supersonic Islands

- Wide islands satisfying

$$W \gg L_s/L_n$$

termed *subsonic* islands. Expect such islands to exhibit flattened density profile within separatrix. Subsonic islands strongly coupled to both electron and ion fluids.

- Narrow islands satisfying

$$W \ll L_s/L_n$$

termed *supersonic* islands. No flattening of density profile within separatrix. Supersonic islands strongly coupled to electron fluid, but only weakly coupled to ion fluid.

Subsonic Islands:[a] Introduction

- To lowest order:

$$\phi = \phi(\psi), \;\; n = n(\psi).$$

- Follows that both electron stream-function, $\phi_e = \phi - n$, and ion stream-function, $\phi_i = \phi + \tau\, n$, are flux-surface functions. Both electron and ion fluid flow constrained to follow flux-surfaces.

- Let

$$M(\psi) = d\phi/d\psi, \;\; L(\psi) = dn/d\psi.$$

- Follows that

$$V_{E\times B\, y} = x\, M, \;\; V_{e\, y} = x\,(M - L), \;\; V_{i\, y} = x\,(M + \tau\, L).$$

[a] R. Fitzpatrick, F.L. Waelbroeck, Phys. Plasmas **12**, 022307 (2005).

Subsonic Islands: Density Flattening

- By symmetry, both $M(\psi)$ and $L(\psi)$ are *odd* functions of x. Hence,

$$M(\psi) = L(\psi) = 0$$

  inside separatrix: *i.e.*, no electron/ion flow within separatrix in island frame.

- Electron/ion fluids constrained to propagate with island inside separatrix.

- Density profile *flattened* within separatrix.

Subsonic Islands: Analysis - I

- Density equation reduces to

$$0 \simeq [V_z + J, \psi] + D\,\nabla^2 n.$$

- Vorticity equation reduces to

$$\begin{aligned} 0 \;&\simeq\; \left[-M\,U - (\tau/2)(L\,U + M\,\nabla^2 n) + J, \psi\right] \\ &\quad + \mu_i\,\nabla^4(\phi + \tau\,n) + \mu_e\,\nabla^4(\phi - n). \end{aligned}$$

- Flux-surface average both equations, recalling that $\langle [A, \psi] \rangle = 0$.

Subsonic Islands: Analysis - II

- Obtain

$$\langle \nabla^2 n \rangle \simeq 0,$$

and

$$(\mu_i + \mu_e)\, \langle \nabla^4 \phi \rangle + (\mu_i\, \tau - \mu_e)\, \langle \nabla^4 n \rangle \simeq 0.$$

- Solution outside separatrix:

$$M(\psi) = -\frac{(\mu_i\, \tau - \mu_e)}{(\mu_i + \mu_e)}\, L(\psi) + F(\psi),$$

where

$$L(\psi) = -\mathrm{sgn}(x)\, L_0/\langle x^2 \rangle,$$

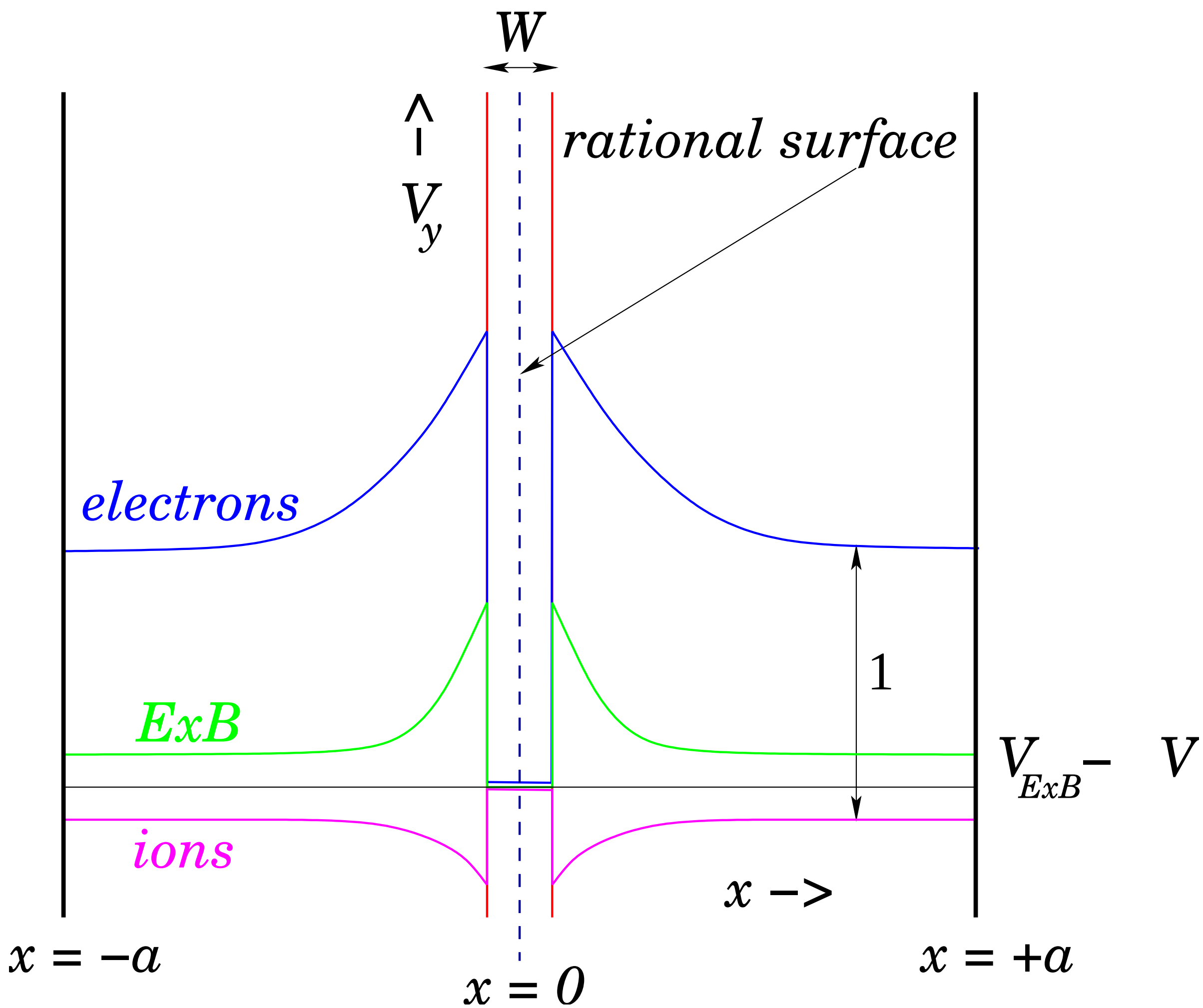
and $F(\psi)$ is previously obtained MHD profile:

$$F(\psi) = \mathrm{sgn}(x)\, F_0\, \int_{-\Psi}^{\psi} d\psi/\langle x^4 \rangle \bigg/ \int_{-\Psi}^{-\infty} d\psi/\langle x^4 \rangle.$$

Subsonic Islands: Velocity Profiles

- As $|x|/W \to \infty$ then $x\,L \to L_0$ and $x\,F \to |x|\,F_0$.
- $L(\psi)$ corresponds to *localized* velocity profile. $F(\psi)$ corresponds to *non-localized* profile. Require localized profile, so $F_0 = 0$.
- Velocity profiles outside separatrix (using b.c. on n):

$$
\begin{aligned}
V_{y\,i} &\simeq +\frac{\mu_e}{\mu_i+\mu_e}\,\frac{|x|}{\langle x^2\rangle},\\
V_{y\,E\times B} &\simeq -\frac{(\mu_i\,\tau-\mu_e)}{(1+\tau)\,(\mu_i+\mu_e)}\,\frac{|x|}{\langle x^2\rangle},\\
V_{y\,e} &= -\frac{\mu_i}{\mu_i+\mu_e}\,\frac{|x|}{\langle x^2\rangle}.
\end{aligned}
$$


W
$\hat{V}_y$
rational surface
electrons
ExB
1
V_{ExB} − V
ions
x −>
x = −a
x = 0
x = +a

Subsonic Islands: Island Propagation

- As $|x|/W \to \infty$ expect $V_{y\,E\times B} \to V_{EB} - V$, where V_{EB} is unperturbed (*i.e.*, no island) $\vec{E}\times\vec{B}$ velocity at rational surface (in lab. frame), and V is island phase-velocity (in lab. frame).

- Hence
$$V = V_{EB} + \frac{(\mu_i\,\tau - \mu_e)}{(1+\tau)\,(\mu_i+\mu_e)}.$$

- But unperturbed ion/electron fluid velocities (in lab. frame):
$$V_i = V_{EB} + \tau/(1+\tau), \quad V_e = V_{EB} - 1/(1+\tau).$$

- Hence
$$V = \frac{\mu_i}{\mu_i+\mu_e}\,V_i + \frac{\mu_e}{\mu_i+\mu_e}\,V_e.$$
So, island phase-velocity is *viscosity weighted average* of unperturbed ion/electron fluid velocities.

Subsonic Islands: Polarization Term - I

- Vorticity equation yields

$$J_c \simeq \frac{1}{2}\left(x^2 - \frac{\langle x^2\rangle}{\langle 1\rangle}\right)\frac{d[M\,(M+\tau\,L)]}{d\psi} + I(\psi)$$

outside separatrix, where J_c is part of J with $\cos\theta$ symmetry.

- As before, flux-surface average of Ohm's law yields:

$$\langle J_c\rangle = I(\psi)\langle 1\rangle = \eta^{-1}\frac{d\Psi}{dt}\langle\cos\theta\rangle.$$

- Hence

$$J_c \simeq \frac{1}{2}\left(x^2 - \frac{\langle x^2\rangle}{\langle 1\rangle}\right)\frac{d[M\,(M+\tau\,L)]}{d\psi} + \eta^{-1}\frac{d\Psi}{dt}\frac{\langle\cos\theta\rangle}{\langle 1\rangle}.$$

Subsonic Islands: Polarization Term - II

- Asymptotic matching between inner and outer regions yields:

$$\Delta' \Psi = -4 \int_{+\Psi}^{-\infty} \langle J_c \, \cos\theta \rangle \, d\psi.$$

- Evaluating flux-surface integrals, making use of previous solutions for M and L, obtain modified Rutherford equation:

$$\frac{0.823}{\eta} \frac{dW}{dt} \simeq \Delta' + 1.38 \, \beta \, \frac{(V - V_{EB})\,(V - V_i)}{(W/4)^3}.$$

- New term is due to *polarization current* associated with ion fluid flow around curved island flux-surfaces (in island frame). Obviously, new term is zero if island propagates with ion fluid: *i.e.*, $V = V_i$.

Subsonic Islands: Summary

- Results limited to large islands: *i.e.*, large enough for sound waves to flatten density profile.
- Island propagates at (perpendicular) viscosity weighted average of unperturbed (no island) ion and electron fluid velocities.
- Polarization term in Rutherford equation is stabilizing provided ion (perpendicular) viscosity greatly exceeds electron (perpendicular) viscosity (which is what we expect), and destabilizing otherwise.