Introduction to Magnetic Island Theory [a]

Richard Fitzpatrick

Institute for Fusion Studies

University of Texas at Austin

Austin, TX, USA

[a] Lectures based on work of R. Fitzpatrick, F.L. Waelbroeck, and F. Militello.

Outline

Lecture 1: Introduction. MHD theory.

Lecture 2: Neoclassical effects. Drift-MHD theory. Subsonic islands.

Lecture 3: Supersonic islands. Further work.

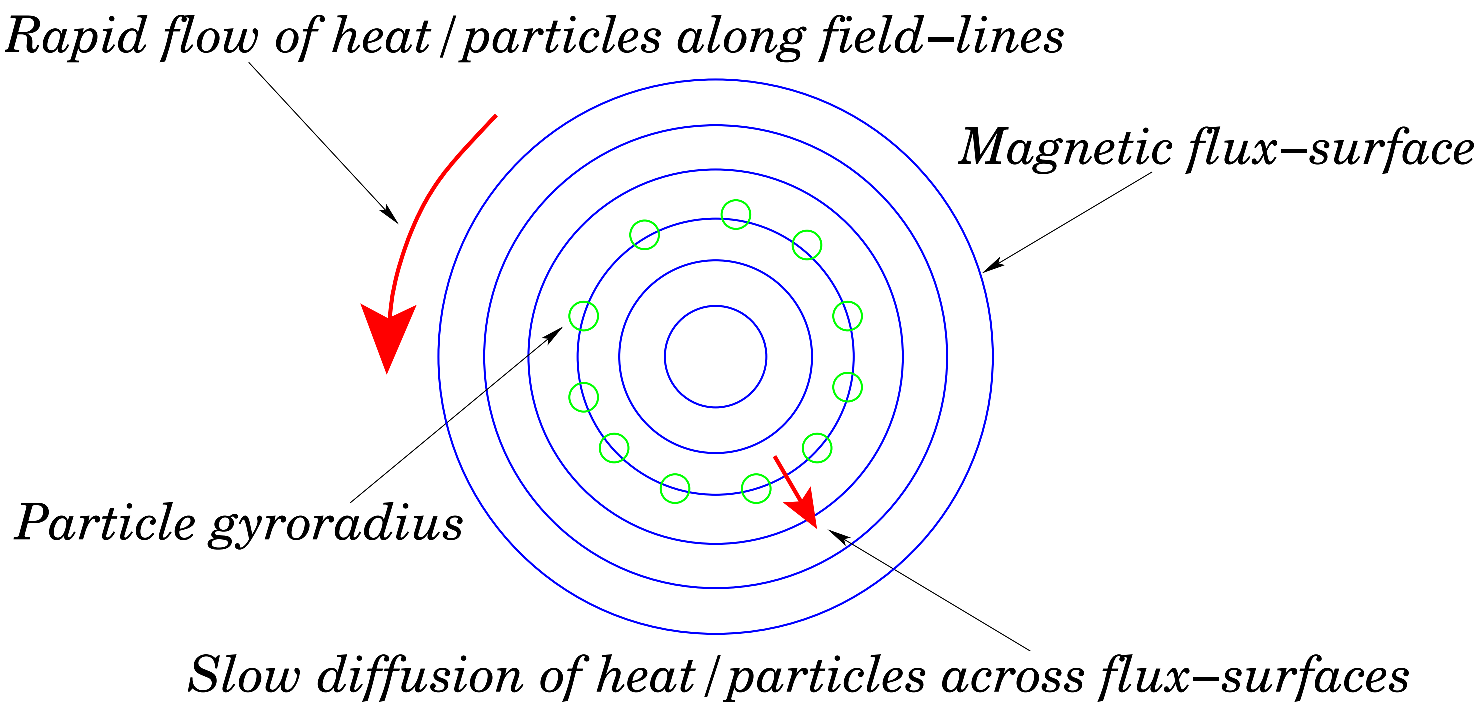
Lecture 1

Introduction: Toroidal Magnetic Confinement

- Toroidal magnetic confinement devices designed to trap hot plasma within set of toroidally nested magnetic flux-surfaces.[a]
- Basic idea—charged particles free to stream along field-lines, but "stick" to magnetic flux-surfaces due to their (relatively) small gyroradii.
- Heat/particles flow rapidly along field-lines, but can only diffuse relatively slowly across flux-surfaces. Diffusion rate controlled by small-scale plasma turbulence.

[a] *Tokamaks*, 3rd Edition, J. Wesson (Oxford University Press, 2004). *Ideal Magnetohydrodynamics*, J.P. Freidberg (Springer, 1987).

Poloidal Cross–Section of Toroidal Confinement Device

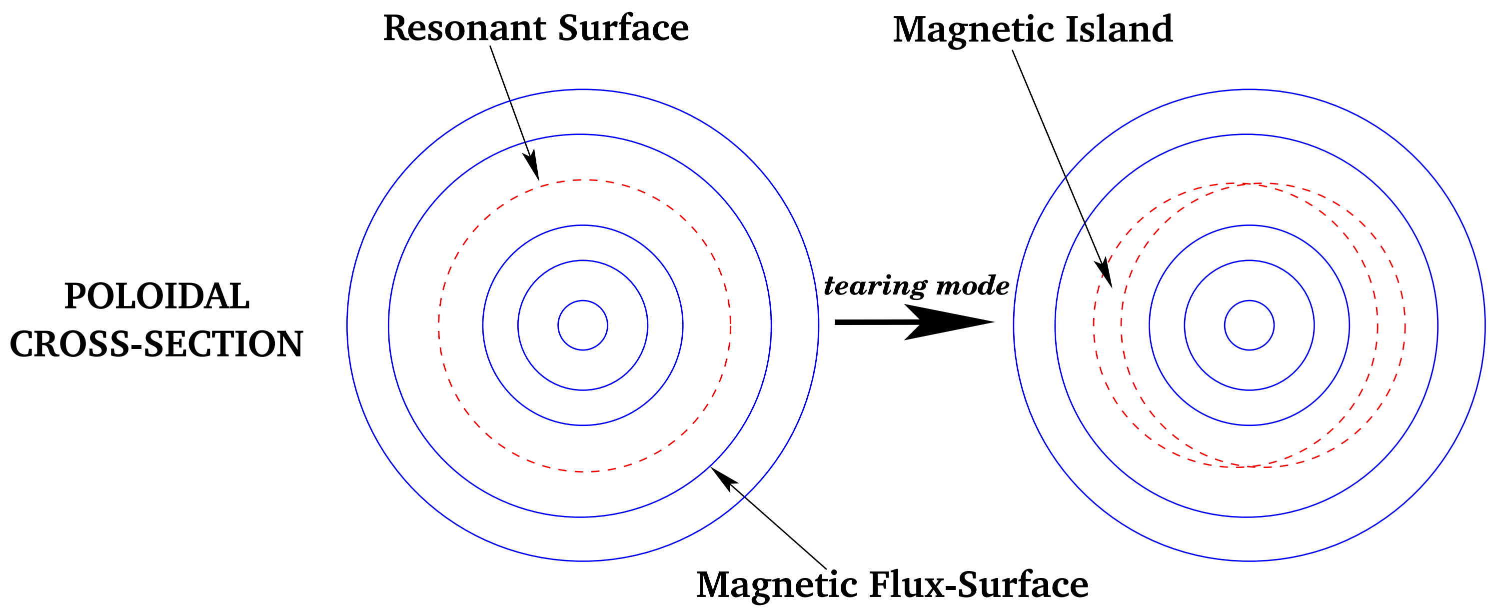


Introduction: Macroscopic Instabilities

- Two main types of macroscopic instabilities[a] in toroidal magnetic confinement devices:
  - Catastrophic "ideal" (*i.e.*, non-reconnecting) instabilities, which destroy plasma in matter of micro-seconds—we know how to avoid these.
  - Slowly growing "tearing" instabilities, which reconnect magnetic flux-surfaces to form *magnetic islands*, thereby degrading their confinement properties—much harder to avoid.

[a] *MHD Instabilities*, G. Bateman (MIT, 1978).

Introduction: Magnetic Islands



- Centered on *rational flux-surfaces* which satisfy $\vec{k} \cdot \vec{B} = 0$, where $\vec{k}$ is wave-number of mode, and $\vec{B}$ is equilibrium magnetic field.
- Effectively "short-circuit" confinement by allowing heat/particles to transit island region by rapidly flowing along field-lines, rather than slowly diffusing across flux-surfaces.

Introduction: Need for Magnetic Island Theory

- Magnetic island formation associated with *nonlinear* phase of tearing mode growth (*i.e.*, when island width becomes greater than linear layer width at rational surface).

- In very hot plasmas found in modern-day magnetic confinement devices, linear layers so thin that tearing mode already in nonlinear regime when first detected.

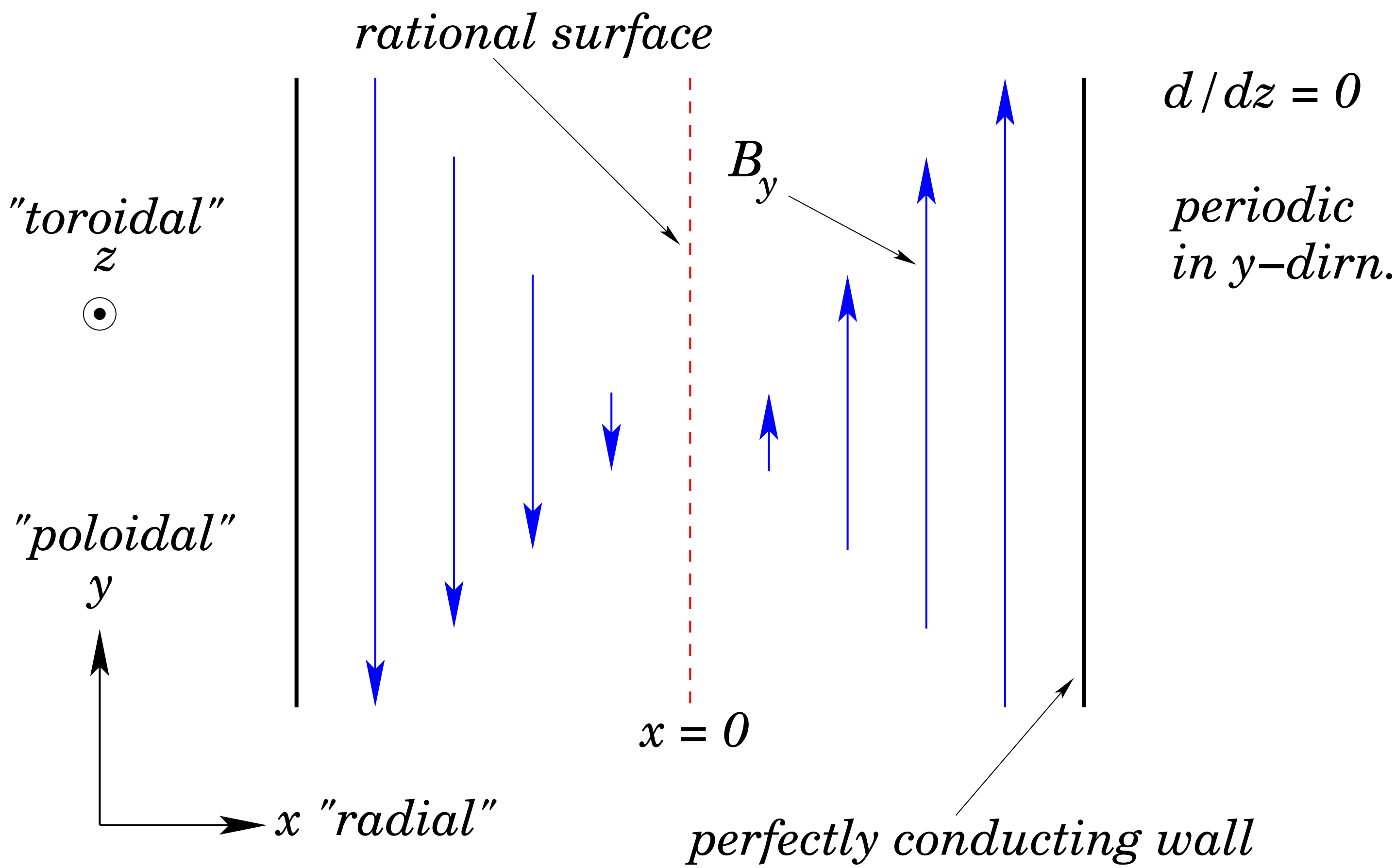
- Linear tearing mode theory largely irrelevant. Require nonlinear magnetic island theory to explain experimental observations.

MHD Theory: Introduction

- Tearing modes are macroscopic instabilities which affect whole plasma. Natural to investigate them using some form of *fluid-theory*.
- Simplest fluid theory is well-known *magnetohydrodynamical approximation*,[a] which effectively treats plasma as *single-fluid*.
- Shall also use *slab approximation* to simplify analysis.

[a] *Plasma Confinement*, R.D. Hazeltine, and J.D. Meiss (Dover, 2003).

Slab Approximation



MHD Theory: Slab Model

- Cartesian coordinates: $(x,\, y,\, z)$. Let $\partial/\partial z \equiv 0$.
- Assume presence of dominant uniform "guide-field" $\vec{B}_z\,\vec{z}$.
- All field-strengths normalized to B_z.
- All lengths normalized to equilibrium magnetic shear-length:

$$L_s = B_z/B_y'(0).$$

- All times normalized to shear-Alfvén time calculated with B_z.
- Perfect wall boundary conditions at $x = \pm a$.
- Wave-number of tearing instability: $\vec{k} = (0, k, 0)$, so $\vec{k}\cdot\vec{B} = 0$ at $x = 0$. Hence, rational surface at $x = 0$.

MHD Theory: Model MHD equations

- Let $\vec{B}_\perp = \nabla\psi \times \vec{z}$ and $\vec{V} = \nabla\phi \times \vec{z}$, where $\vec{V}$ is $\vec{E} \times \vec{B}$ velocity.
- $\vec{B} \cdot \nabla\psi = \vec{V} \cdot \nabla\phi = 0$, so ψ maps magnetic flux-surfaces, and ϕ maps stream-lines of $\vec{E} \times \vec{B}$ fluid.
- Incompressible MHD equations: [a]

$$\begin{aligned} \frac{\partial\psi}{\partial t} &= [\phi, \psi] + \eta\, J, \\ \frac{\partial U}{\partial t} &= [\phi, U] + [J, \psi] + \mu\, \nabla^2 U, \end{aligned}$$

  where $J = \nabla^2\psi$, $U = \nabla^2\phi$, and $[A, B] = A_x\, B_y - A_y\, B_x$. Here, η is resistivity, and μ is viscosity. In normalized units: $\eta, \mu \ll 1$.

- First equation is z-component of Ohm's law. Second equation is z-component of curl of plasma equation of motion.

[a] *Plasma Confinement*, R.D. Hazeltine, and J.D. Meiss (Dover, 2003).

MHD Theory: Outer Region

- In "outer region", which comprises most of plasma, non-linear, non-ideal (η and μ), and inertial effects ($\partial/\partial t$ and $\vec{V}\cdot\nabla$), negligible.

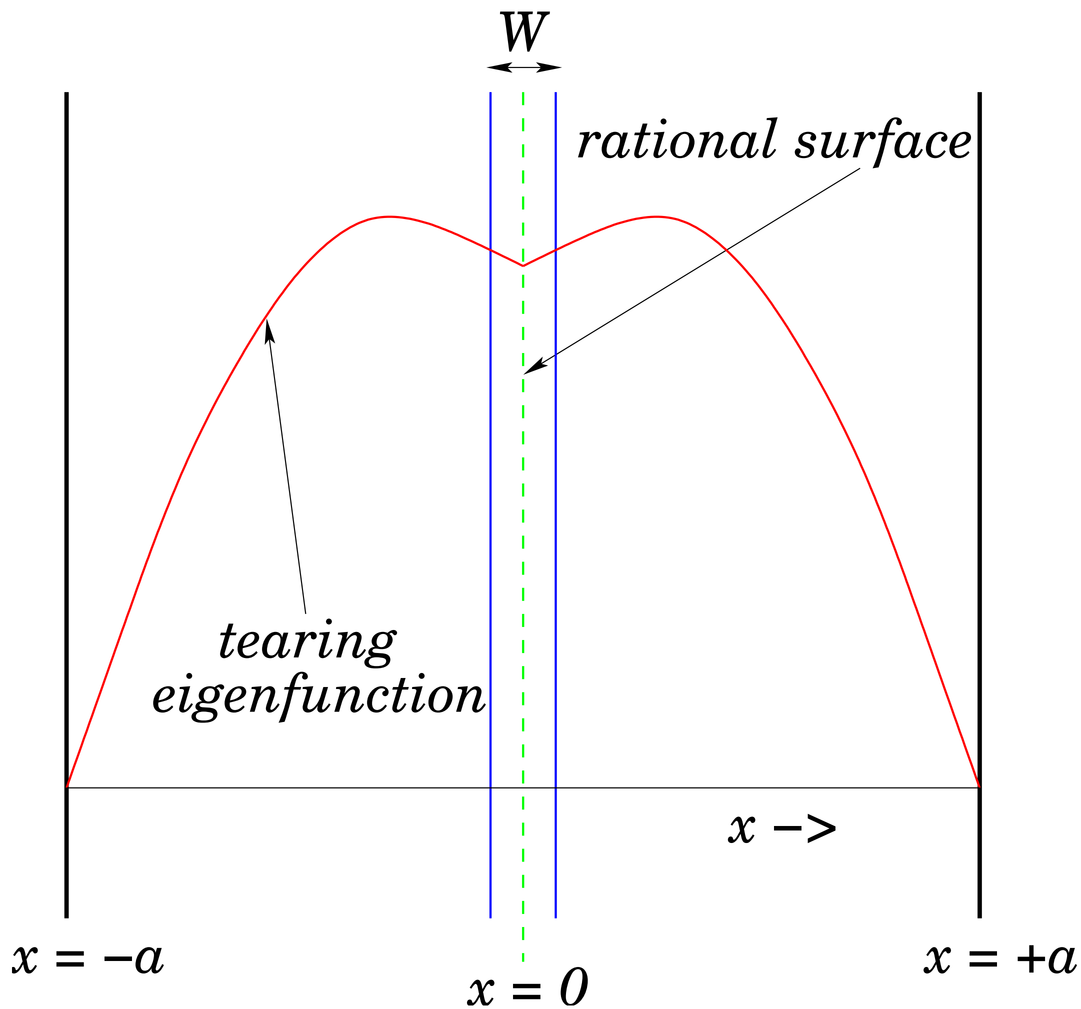
- Vorticity equation reduces to

$$[J,\psi] \simeq 0.$$

- When linearized, obtain $\psi(x,y) = \psi^{(0)}(x) + \psi^{(1)}(x)\,\cos(k\,y)$, where $B_y^{(0)} = -d\psi^{(0)}/dx$, and

$$\left(\frac{d^2}{dx^2} - k^2\right)\psi^{(1)} - \left(\frac{d^2 B_y^{(0)}/dx^2}{B_y^{(0)}}\right)\psi^{(1)} = 0.$$

- Equation is *singular* at rational surface, $x = 0$, where $B_y^{(0)} = 0$.



W
rational surface
tearing
eigenfunction
x ->
x = −a
x = 0
x = +a

MHD Theory: Tearing Stability Index

- Find tearing eigenfunction, $\psi^{(1)}(x)$, which is continuous, has tearing parity $[\psi^{(1)}(-x) = \psi^{(1)}(x)]$, and satisfies boundary condition $\psi^{(1)}(a) = 0$ at conducting wall.

- In general, eigenfunction has *gradient discontinuity* across rational surface (at $x = 0$). Allowed because tearing mode equation singular at rational surface.

- Tearing stability index:

$$\Delta' = \left[\frac{d\ln\psi^{(1)}}{dx}\right]_{0-}^{0+}.$$

- According to conventional MHD theory,[a] tearing mode is unstable if $\Delta' > 0$.

[a] H.P. Furth, J. Killeen, and M.N. Rosenbluth, Phys. Fluids **6**, 459 (1963).

MHD Theory: Inner Region

- "Inner region" centered on rational surface, $x = 0$. Of extent, $W \ll 1$, where W is magnetic island width (in x).
- In inner region, non-ideal effects, non-linear effects, and plasma inertia can all be important.
- Inner solution must be asymptotically matched to outer solution already obtained.

MHD Theory: Constant-ψ Approximation

- $\psi^{(1)}(x)$ generally does not vary significantly in x over inner region:

$$|\psi^{(1)}(W) - \psi^{(1)}(0)| \ll |\psi^{1}(0)|.$$

- *Constant-ψ approximation*: treat $\psi^{(1)}(x)$ as constant in x over inner region.

- Approximation valid provided

$$|\Delta'|\,W \ll 1,$$

which is easily satisfied for conventional tearing modes.

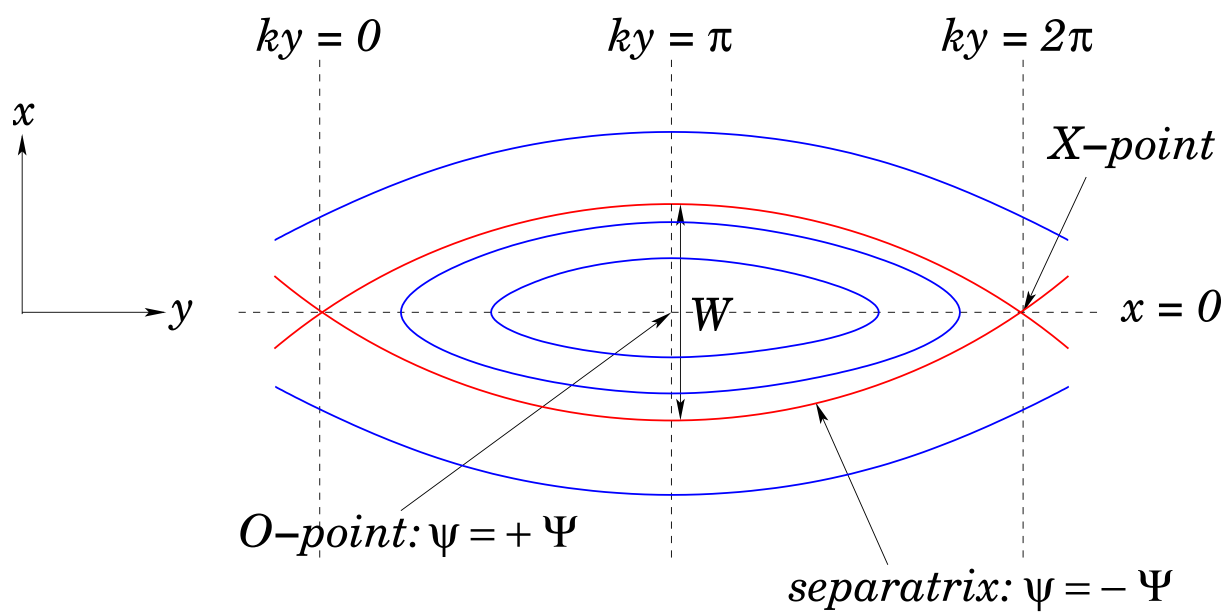
MHD Theory: Constant-ψ Magnetic Island

- In vicinity of rational surface, $\psi^{(0)} \to -x^2/2$, so

$$\psi(x, y, t) \simeq -x^2/2 + \Psi(t)\,\cos\theta,$$

where $\Psi = \psi^{(1)}(0)$ is "reconnected flux", and $\theta = k\,y$.

- Full island width, $W = 4\sqrt{\Psi}$.



MHD Theory: Flux-Surface Average Operator

- Flux-surface average operator is annihilator of Poisson bracket $[A, \psi] \equiv \vec{B} \cdot \nabla A \equiv k\, x\, (\partial A/\partial\theta)_\psi$ for any A: *i.e.*,

$$\langle [A, \psi] \rangle \equiv 0.$$

- Outside separatrix:

$$\langle f(\psi, \theta) \rangle = \oint \frac{f(\psi, \theta)}{|x|} \, \frac{d\theta}{2\pi}.$$

- Inside separatrix:

$$\langle f(s, \psi, \theta) \rangle = \int_{-\theta_0}^{\theta_0} \frac{f(s, \psi, \theta) + f(-s, \psi, \theta)}{2\,|x|} \, \frac{d\theta}{2\pi},$$

where $s = \mathrm{sgn}(x)$, and $x(s, \psi, \theta_0) = 0$.

MHD Theory: MHD Flow -I

- Move to island frame. Look for steady-state solution: $\partial/\partial t = 0$.[a]
- Ohm's law:

$$0 \simeq [\phi, \psi] + \eta\, J.$$

- Since $\eta \ll 1$, first term potentially much larger than second.
- To lowest order:

$$[\phi, \psi] \simeq 0.$$

- Follows that

$$\phi = \phi(\psi):$$

i.e., MHD flow constrained to be around flux-surfaces.

[a] F.L. Waelbroeck, and R. Fitzpatrick, Phys. Rev. Lett. **78**, 1703 (1997).

MHD Theory: MHD Flow - II

- Let
$$M(\psi) = \frac{d\phi}{d\psi}.$$
- Easily shown that
$$V_y = x\,M.$$
- By symmetry, $M(\psi)$ is *odd* function of x. Hence,
$$M = 0$$
inside separatrix: *i.e.*, no flow inside separatrix in island frame. Plasma *trapped* within magnetic separatrix.

MHD Theory: MHD Flow - III

- Vorticity equation:

$$0 \simeq [-M\,U + J, \psi] + \mu\,\nabla^4 \phi.$$

- Flux-surface average, recalling that $\langle [A, \psi] \rangle = 0$:

$$\langle \nabla^4 \phi \rangle \equiv -\frac{d^2}{d\psi^2}\left(\langle x^4 \rangle \frac{dM}{d\psi}\right) \simeq 0.$$

- Solution outside separatrix:

$$M(\psi) = \mathrm{sgn}(x)\, M_0 \int_{-\Psi}^{\psi} d\psi / \langle x^4 \rangle \bigg/ \int_{-\Psi}^{-\infty} d\psi / \langle x^4 \rangle.$$

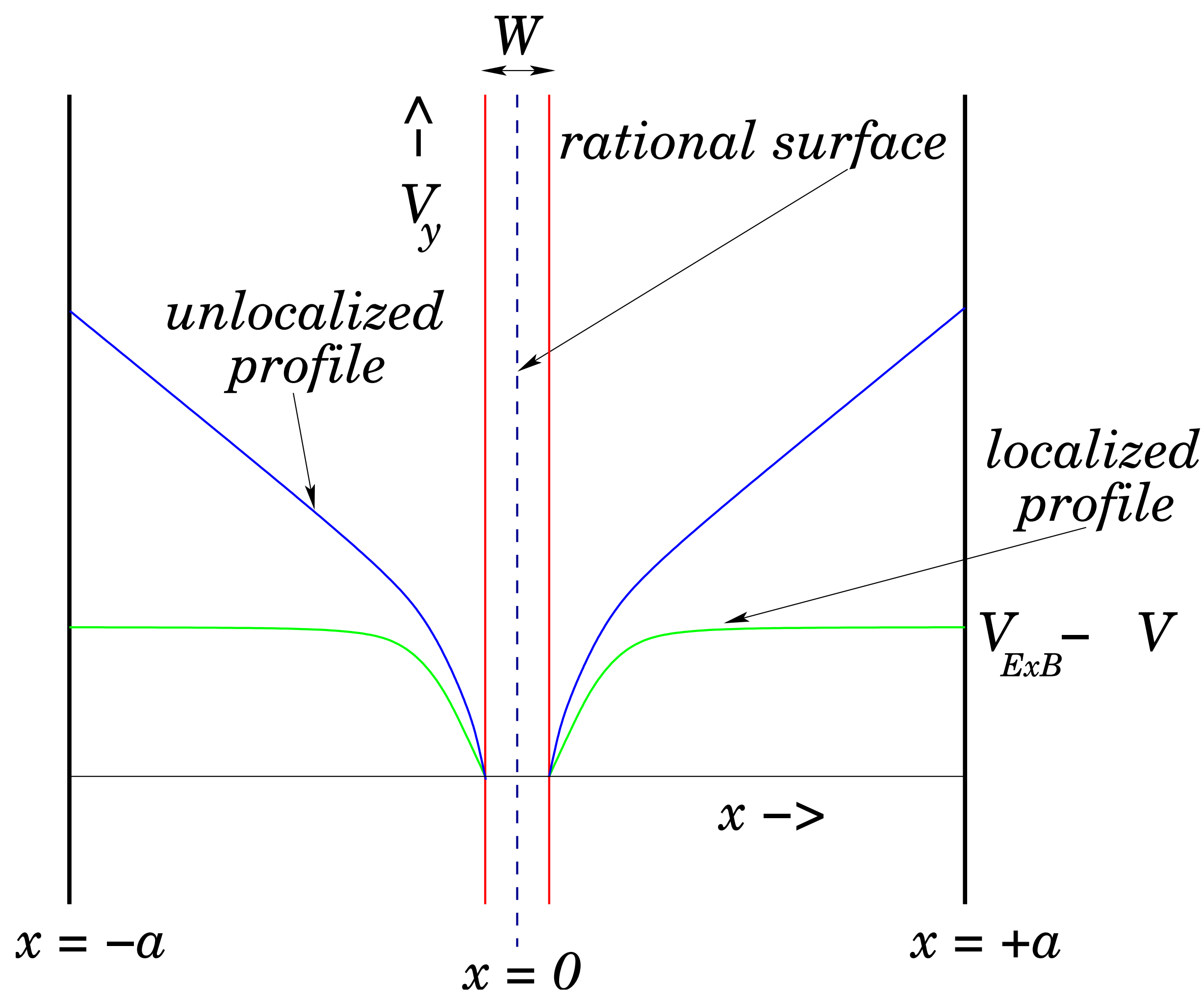
MHD Theory: MHD Flow - IV

- Note
$$V_y = x\, M \rightarrow |x|\, M_0$$
as $|x|/W \rightarrow \infty$.

- V-shaped velocity profile which extends over whole plasma.

- Expect *isolated* magnetic island to have *localized* velocity profile. Suggests that $M_0 = 0$ for isolated island.

- Hence, zero MHD flow in island frame: *i.e.*, island propagates at local $\vec{E} \times \vec{B}$ velocity.


W
^
|
V_y
rational surface
unlocalized
profile
localized
profile
$V_{ExB} - V$
x ->
$x = -a$
$x = 0$
$x = +a$

MHD Theory: Rutherford Equation - I

- Asymptotic matching between inner and outer regions yields:

$$\Delta' \Psi = -4 \int_{+\Psi}^{-\infty} \langle J \cos\theta \rangle \, d\psi.$$

- In island frame, in absence of MHD flow, vorticity equation reduces to

$$[J, \psi] \simeq 0.$$

- Hence,

$$J = J(\psi).$$

MHD Theory: Rutherford Equation - II

- Ohm's law:

$$\frac{d\Psi}{dt}\cos\theta \simeq [\phi,\psi] + \eta\, J(\psi).$$

- Have shown there is no MHD-flow [*i.e.*, $\phi \sim O(1)$], but can still be *resistive flow* [*i.e.*, $\phi \sim O(\eta)$].

- Eliminate resistive flow by flux-surface averaging:

$$\frac{d\Psi}{dt}\,\langle\cos\theta\rangle \simeq \eta\, J(\psi)\,\langle 1\rangle.$$

- Hence,

$$\Delta'\,\Psi \simeq -\frac{4}{\eta}\,\frac{d\Psi}{dt}\int_{+\Psi}^{-\infty}\frac{\langle\cos\theta\rangle^2}{\langle 1\rangle}\,d\psi.$$

MHD Theory: Rutherford Equation - III

- Use $W = 4\sqrt{\Psi}$, and evaluate integral. Obtain *Rutherford island width evolution equation*:[a]

$$\frac{0.823}{\eta}\frac{dW}{dt} \simeq \Delta'.$$

- According to Rutherford equation, island grows *algebraically* on *resistive time-scale*.

- Rutherford equation does not predict island saturation.

[a] P.H. Rutherford, Phys. Fluids **16**, 1903 (1973).

MHD Theory: Rutherford Equation - IV

- Higher order asymptotic matching between inner and outer regions yields:[a]

$$\frac{0.823}{\eta}\frac{dW}{dt} \simeq \Delta' - 0.41 \left(-\frac{d^4 B_y^{(0)}/dx^4}{d^2 B_y^{(0)}/dx^2} \right)_{x=0} W.$$

- Hence, saturated ($d/dt = 0$) island width is

$$W_0 = \frac{\Delta'}{0.41} \left(-\frac{d^2 B_y^{(0)}/dx^2}{d^4 B_y^{(0)}/dx^4} \right)_{x=0}.$$

[a] F. Militello, and F. Porcelli, Phys. Plasmas **11**, L13 (2004). D.F. Escande, and M. Ottaviani, Physics Lett. A **323**, 278 (2004).

MHD Theory: Summary

- Tearing mode unstable if $\Delta' > 0$.
- Island propagates at local $\vec{E} \times \vec{B}$ velocity at rational surface.
- Island grows algebraically on resistive time-scale.
- Saturated island width:

$$W_0 = \frac{\Delta'}{0.41}\left(-\frac{d^2 B_y^{(0)}/dx^2}{d^4 B_y^{(0)}/dx^4}\right)_{x=0}.$$