

# Magnetic Reconnection in Tokamaks

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## Outline

1. Introduction.
2. Toroidal magnetic confinement.
3. MHD theory.
4. Profile modification.
5. Bootstrap current destabilization.
6. Drift-MHD theory.
7. Subsonic island theory.
8. Supersonic island theory.
9. Other effects.

# 1: Introduction

## Magnetic Reconnection in Astrophysical Plasmas

- Narrow current sheets. Highly unstable to tearing instabilities.
- Very rapid (quasi-Alfvénic) reconnection rates.
- Reconnection gives rise to significant release of thermal energy, as well as copious charged particle acceleration.
- Main aims of astrophysical reconnection theory are to explain rapid onset of reconnection, to account for fast reconnection rate, and to understand energy release and particle acceleration mechanisms.

## Magnetic Reconnection<sup>a</sup> in Tokamak Plasmas

- Extended current distributions. Comparatively stable to tearing instabilities.
- Reconnection changes topology of magnetic flux-surfaces, thereby degrading energy and particle confinement.
- Reconnection-induced energy release and charged particle acceleration completely negligible.
- Reconnection timescale irrelevant since sufficient time for tearing instability growing at smallest conceivable rate to eventually cause significant change in topology of flux-surfaces.
- Main aims of fusion reconnection theory are to understand various factors that determine stability, and final saturated amplitude, of tearing instabilities.

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<sup>a</sup>Excluding sawtooth oscillation.

## 2: Toroidal Magnetic Confinement

## Principles of Tokamak Confinement

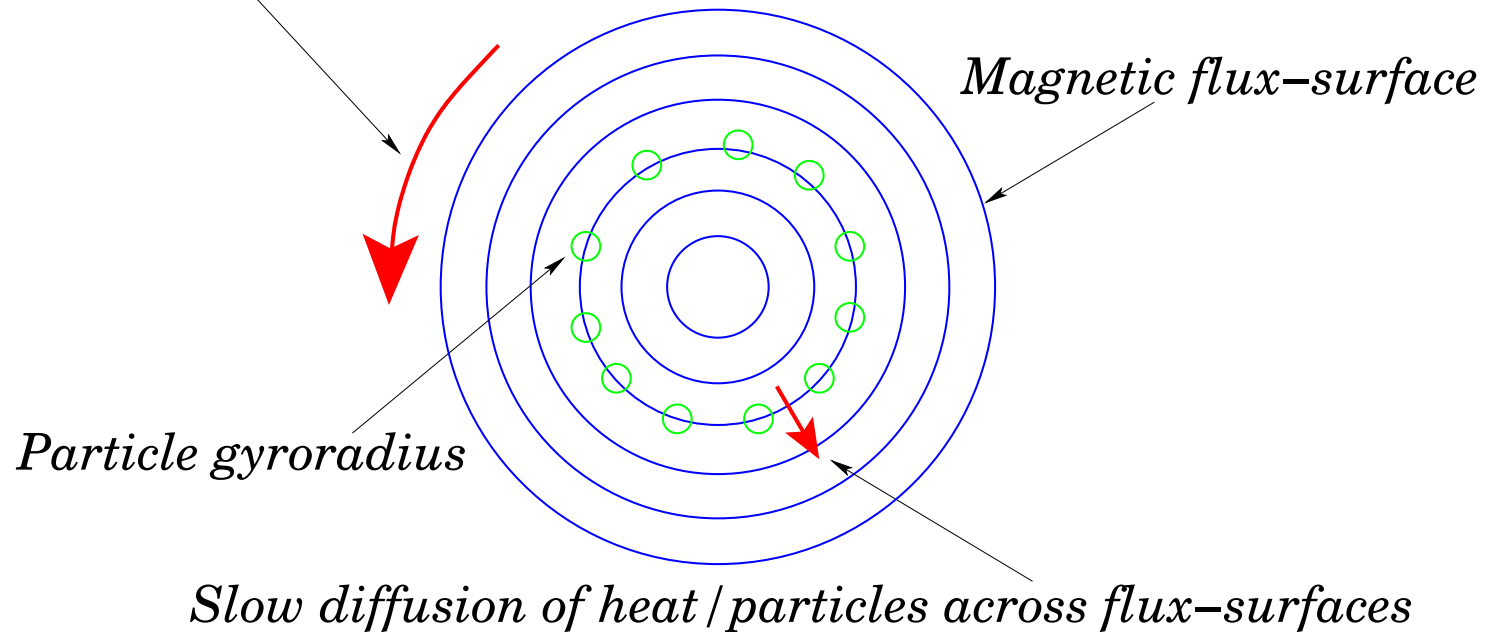
- Tokamaks designed to trap hot plasma on set of axisymmetric, nested, toroidal magnetic flux-surfaces.<sup>a</sup>
- Fundamental principle—charged particles free to stream along magnetic field-lines, but “stick” to flux-surfaces due to their (relatively) small gyroradii.
- Heat/particles flow rapidly along field-lines, but can only diffuse relatively slowly across flux-surfaces. Diffusion rate controlled by small-scale plasma turbulence.

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<sup>a</sup> *Tokamaks*, 3rd Edition, J. Wesson (Oxford University Press, 2004). *Ideal Magnetohydrodynamics*, J.P. Freidberg (Springer, 1987).

## ***Poloidal Cross-Section of Toroidal Confinement Device***

*Rapid flow of heat / particles along field-lines*



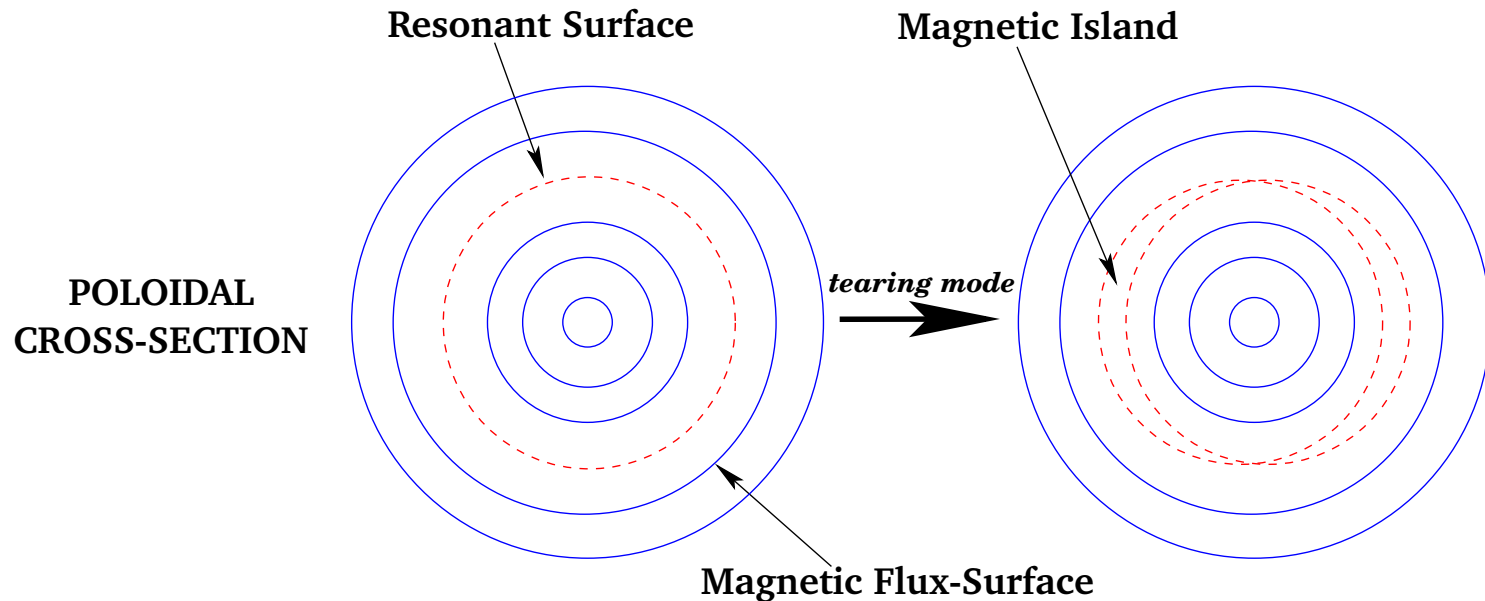
## Macroscopic Instabilities

- Two main types of macroscopic instability<sup>a</sup> in tokamaks:
  - Catastrophic “ideal” (*i.e.*, non-reconnecting) instabilities that disrupt plasma in matter of micro-seconds—easily avoided.
  - Slowly growing “tearing” instabilities that reconnect magnetic flux-surfaces, but eventually saturate at relatively low amplitude to form *magnetic islands*, thereby degrading plasma confinement—much harder to avoid.

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<sup>a</sup>*MHD Instabilities*, G. Bateman (MIT, 1978).

# Magnetic Islands



- Centered on *rational flux-surfaces* which satisfy  $\vec{k} \cdot \vec{B} = 0$ , where  $\vec{k}$  is wave-number of mode, and  $\vec{B}$  is equilibrium magnetic field.
- Effectively “short-circuit” confinement by allowing heat/particles to transit island region by rapidly flowing along field-lines, rather than slowly diffusing across flux-surfaces.

## Need for Magnetic Island Theory

- Magnetic island formation associated with *nonlinear* phase of tearing mode growth (*i.e.*, when island width becomes greater than linear layer width at rational surface).
- In very hot plasmas found in modern-day tokamaks, linear layers so thin that tearing mode already in nonlinear regime when first detected.
- Linear tearing mode theory largely irrelevant. Require nonlinear magnetic island theory to explain experimental observations.

# 3: MHD Theory

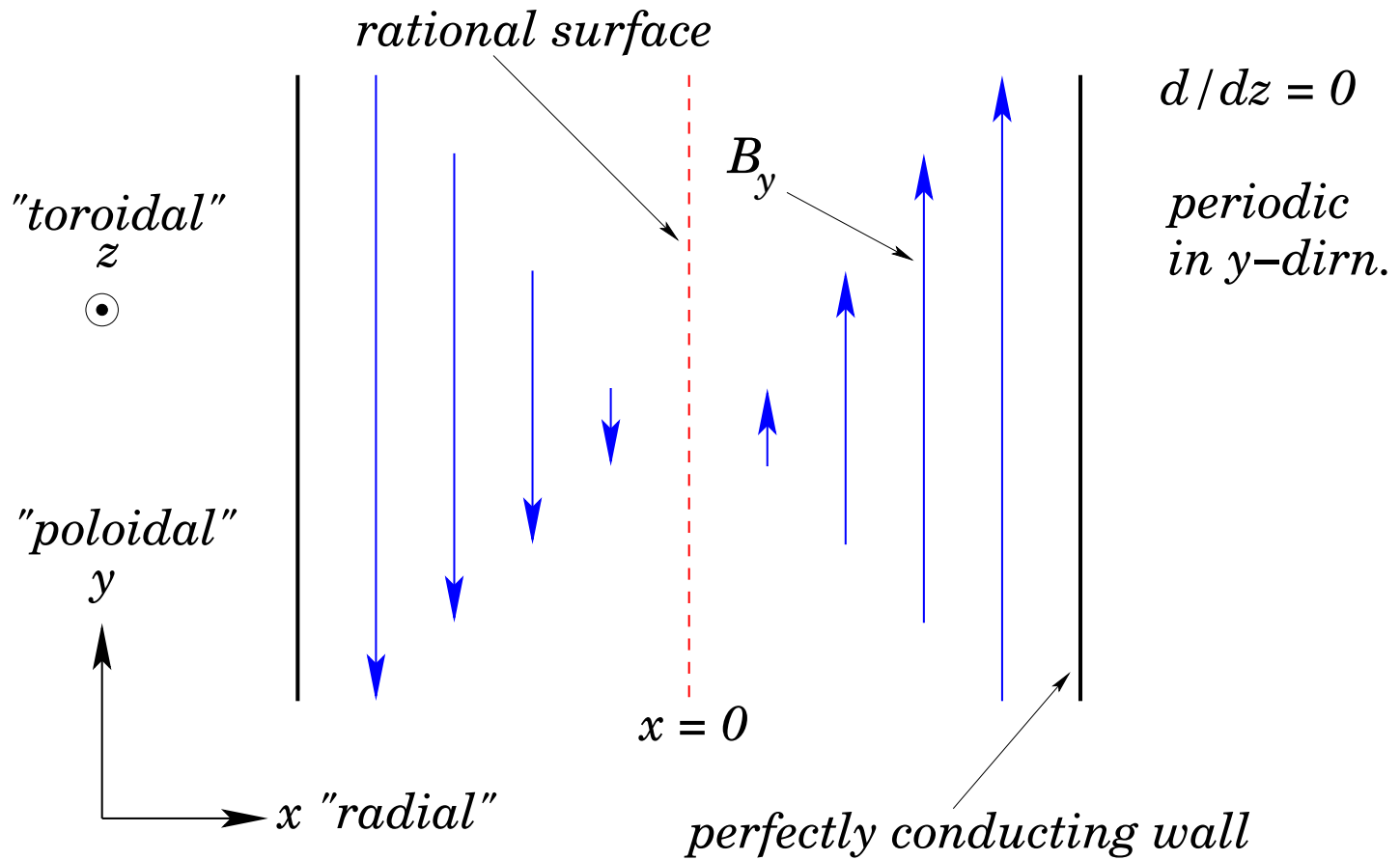
## Introduction

- Tearing modes are macroscopic instabilities that affect whole plasma. Natural to investigate them using some form of *fluid-theory*.
- Simplest fluid theory is well-known *magnetohydrodynamical approximation*,<sup>a</sup> which effectively treats plasma as *single-fluid*.
- Use *slab approximation* to simplify analysis.

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<sup>a</sup>*Plasma Confinement*, R.D. Hazeltine, and J.D. Meiss (Dover, 2003).

## *Slab Approximation*



## Slab Model

- Cartesian coordinates:  $(x, y, z)$ . Let  $\partial/\partial z \equiv 0$ .
- Assume presence of dominant uniform “guide-field”  $\vec{B}_z \vec{e}_z$ .
- All field-strengths normalized to  $B_z$ .
- All lengths normalized to equilibrium magnetic shear-length:

$$L_s = B_z / (dB_y^{(0)} / dx)_{x=0}.$$

- All times normalized to Alfvén time calculated with  $B_z$ .
- Perfect wall boundary conditions at  $x = \pm a$ .
- Wave-number of tearing instability:  $\vec{k} = (0, k, 0)$ , so  $\vec{k} \cdot \vec{B} = 0$  at  $x = 0$ . Hence, rational surface at  $x = 0$ .

## Model MHD Equations

- Let  $\vec{B} = \nabla\psi \times \vec{e}_z + B_z \vec{e}_z$ ,  $\vec{V} = \nabla\phi \times \vec{e}_z$ , where  $\vec{V}$  is  $\vec{E} \times \vec{B}$  vely.
- $\vec{B} \cdot \nabla\psi = \vec{V} \cdot \nabla\phi = 0$ , so  $\psi$  maps magnetic flux-surfaces, and  $\phi$  maps stream-lines of  $\vec{E} \times \vec{B}$  fluid.
- Incompressible MHD equations:<sup>a</sup>

$$\frac{\partial\psi}{\partial t} = [\phi, \psi] + \eta J,$$

$$\frac{\partial\mathcal{U}}{\partial t} = [\phi, \mathcal{U}] + [J, \psi] + \mu \nabla^2 \mathcal{U},$$

where  $J = \nabla^2\psi$ ,  $\mathcal{U} = \nabla^2\phi$ , and  $[A, B] = A_x B_y - A_y B_x$ . Here,  $\eta$  is resistivity, and  $\mu$  is viscosity. In normalized units:  $\eta, \mu \ll 1$ .

- First equation is z-component of Ohm's law. Second equation is z-component of curl of plasma equation of motion.

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<sup>a</sup>*Plasma Confinement*, R.D. Hazeltine, and J.D. Meiss (Dover, 2003).

## Outer Region

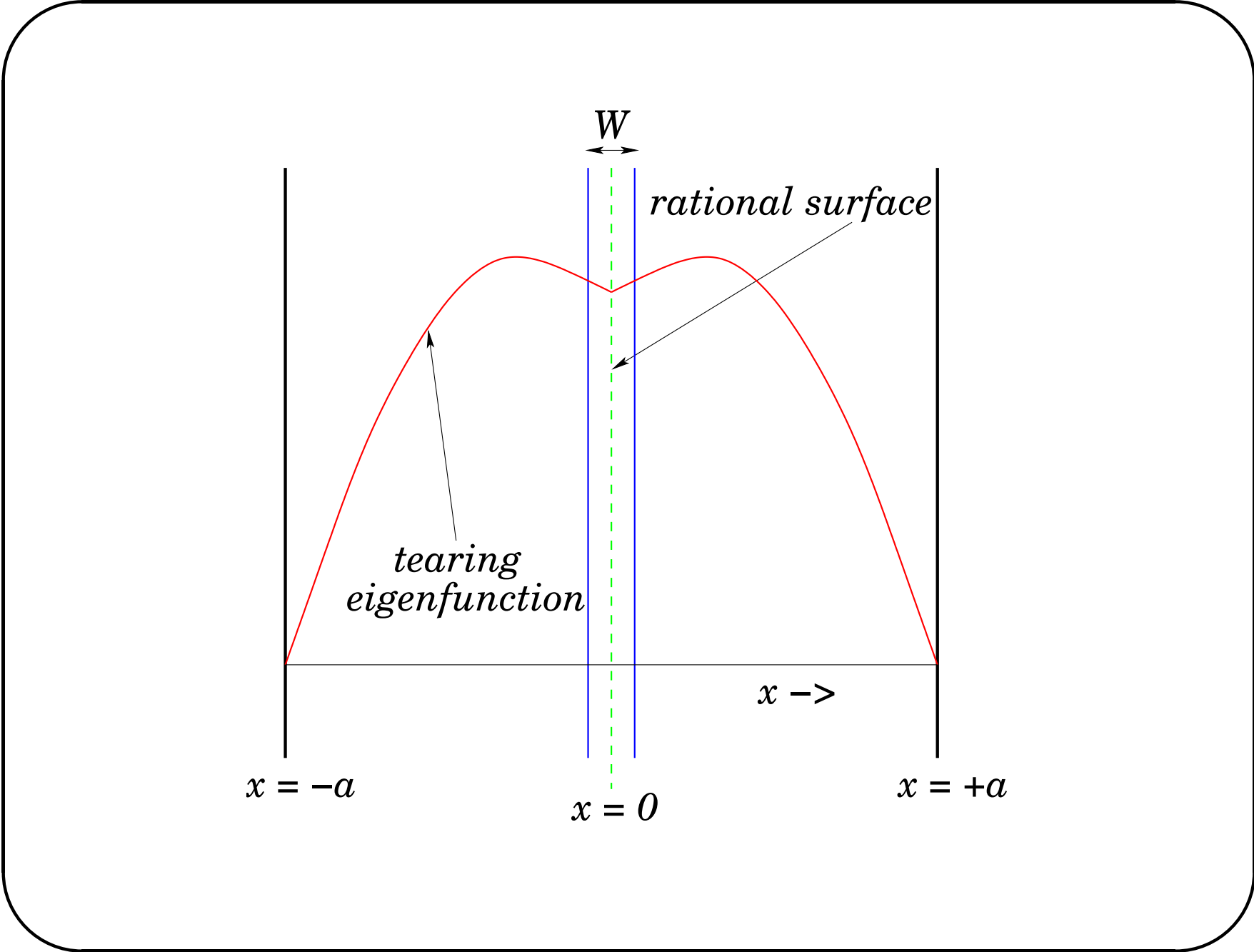
- In “outer region”, which comprises most of plasma, can neglect non-linear, non-ideal ( $\eta$  and  $\mu$ ), and inertial ( $\partial/\partial t$  and  $\vec{V} \cdot \nabla$ ) effects.
- Vorticity equation reduces to

$$[J, \psi] \simeq 0.$$

- When linearized, obtain  $\psi(x, y) = \psi^{(0)}(x) + \psi^{(1)}(x) \cos(ky)$ , where  $B_y^{(0)} = -d\psi^{(0)}/dx$ , and

$$\left( \frac{d^2}{dx^2} - k^2 \right) \psi^{(1)} - \left( \frac{d^2 B_y^{(0)}/dx^2}{B_y^{(0)}} \right) \psi^{(1)} = 0.$$

- Equation is *singular* at rational surface,  $x = 0$ , where  $B_y^{(0)} = 0$ .



## Tearing Stability Index

- Find tearing eigenfunction,  $\psi^{(1)}(x)$ , which is continuous, has tearing parity [ $\psi^{(1)}(-x) = \psi^{(1)}(x)$ ], and satisfies boundary condition  $\psi^{(1)}(a) = 0$  at conducting wall.
- In general, eigenfunction has *gradient discontinuity* across rational surface (at  $x = 0$ ). Allowed because tearing mode equation singular at rational surface.
- Tearing stability index:

$$\Delta' = \left[ \frac{d \ln \psi^{(1)}}{dx} \right]_{0-}^{0+}.$$

- According to conventional MHD theory,<sup>a</sup> tearing mode is unstable if  $\Delta' > 0$ .

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<sup>a</sup>H.P. Furth, J. Killeen, and M.N. Rosenbluth, Phys. Fluids **6**, 459 (1963).

## Inner Region

- “Inner region” centered on rational surface,  $\chi = 0$ . Of extent,  $W \ll 1$ , where  $W$  is magnetic island width (in  $\chi$ ).
- In inner region, non-ideal effects, non-linear effects, and plasma inertia can all be important.
- Inner solution must be asymptotically matched to outer solution already obtained.

## Constant- $\psi$ Approximation

- $\psi^{(1)}(x)$  generally does not vary significantly in  $x$  over inner region:

$$|\psi^{(1)}(W) - \psi^{(1)}(0)| \ll |\psi^{(1)}(0)|.$$

- *Constant- $\psi$  approximation*: treat  $\psi^{(1)}(x)$  as constant in  $x$  over inner region.
- Approximation valid provided

$$|\Delta'| W \ll 1,$$

which is easily satisfied for conventional tearing modes in tokamak plasmas.

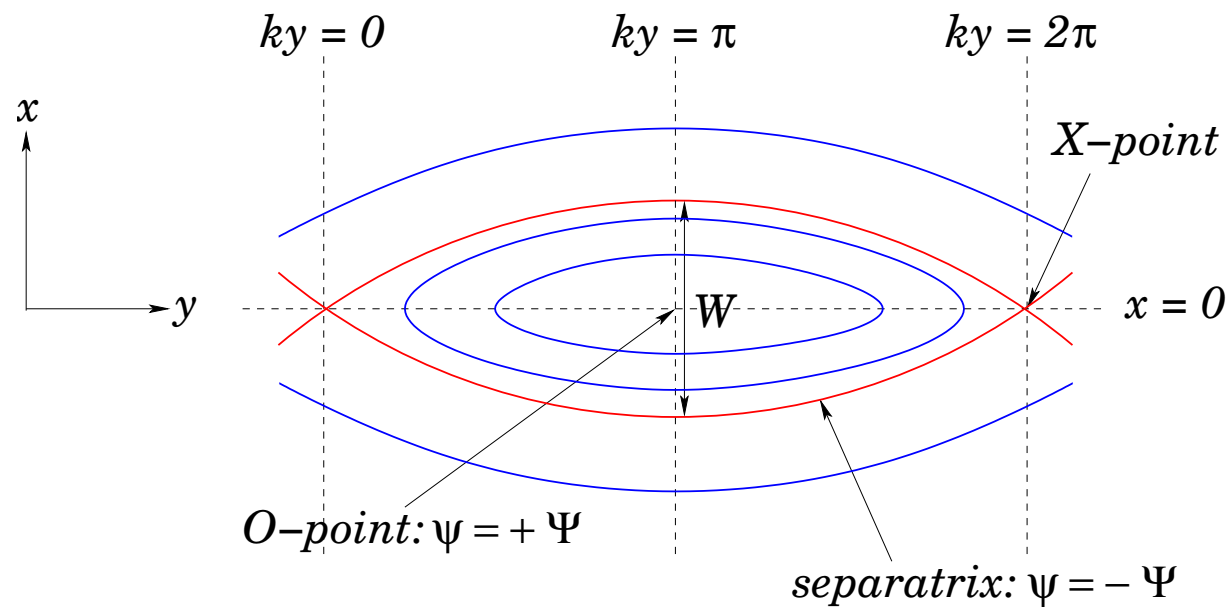
## Constant- $\psi$ Magnetic Island

- In vicinity of rational surface,  $\psi^{(0)} \rightarrow -x^2/2$ , so

$$\psi(x, y, t) \simeq -x^2/2 + \Psi(t) \cos \theta,$$

where  $\Psi = \psi^{(1)}(0)$  is “reconnected flux”, and  $\theta = ky$ .

- Full island width,  $W = 4 \sqrt{\Psi}$ .



## Flux-Surface Average Operator

- Flux-surface average operator is annihilator of Poisson bracket  
 $[A, \psi] \equiv \vec{B} \cdot \nabla A \equiv k \times (\partial A / \partial \theta)_\psi$  for any  $A$ : *i.e.*,

$$\langle [A, \psi] \rangle \equiv 0.$$

- Outside separatrix:

$$\langle f(\psi, \theta) \rangle = \oint \frac{f(\psi, \theta)}{|\mathbf{x}|} \frac{d\theta}{2\pi}.$$

- Inside separatrix:

$$\langle f(s, \psi, \theta) \rangle = \int_{-\theta_0}^{\theta_0} \frac{f(s, \psi, \theta) + f(-s, \psi, \theta)}{2|\mathbf{x}|} \frac{d\theta}{2\pi},$$

where  $s = \text{sgn}(\mathbf{x})$ , and  $\mathbf{x}(s, \psi, \theta_0) = 0$ .

## MHD Flow - I

- Move to island frame. Look for steady-state solution:  $\partial/\partial t = 0$ .<sup>a</sup>
- Ohm's law:

$$0 \simeq [\phi, \psi] + \eta J.$$

- Since  $\eta \ll 1$ , first term potentially much larger than second.
- To lowest order:

$$[\phi, \psi] \simeq 0.$$

- Follows that

$$\phi = \phi(\psi) :$$

*i.e.*, MHD flow constrained to be around flux-surfaces.

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<sup>a</sup>F.L. Waelbroeck, and R. Fitzpatrick, Phys. Rev. Lett. **78**, 1703 (1997).

## MHD Flow - II

- Let

$$M(\psi) = \frac{d\phi}{d\psi}.$$

- Easily shown that

$$V_y = x M.$$

- By symmetry,  $M(\psi)$  is *odd* function of  $x$ . Hence,

$$M = 0$$

inside separatrix: *i.e.*, no flow inside separatrix in island frame.  
Plasma *trapped* within magnetic separatrix.

## MHD Flow - III

- Vorticity equation:

$$0 \simeq [-M \mathbf{U} + \mathbf{J}, \psi] + \mu \nabla^4 \phi.$$

- Flux-surface average, recalling that  $\langle [A, \psi] \rangle = 0$ :

$$\langle \nabla^4 \phi \rangle \equiv -\frac{d^2}{d\psi^2} \left( \langle x^4 \rangle \frac{dM}{d\psi} \right) \simeq 0.$$

- Solution outside separatrix:

$$M(\psi) = \text{sgn}(x) M_0 \int_{-\Psi}^{\psi} d\psi / \langle x^4 \rangle \bigg/ \int_{-\Psi}^{-\infty} d\psi / \langle x^4 \rangle.$$

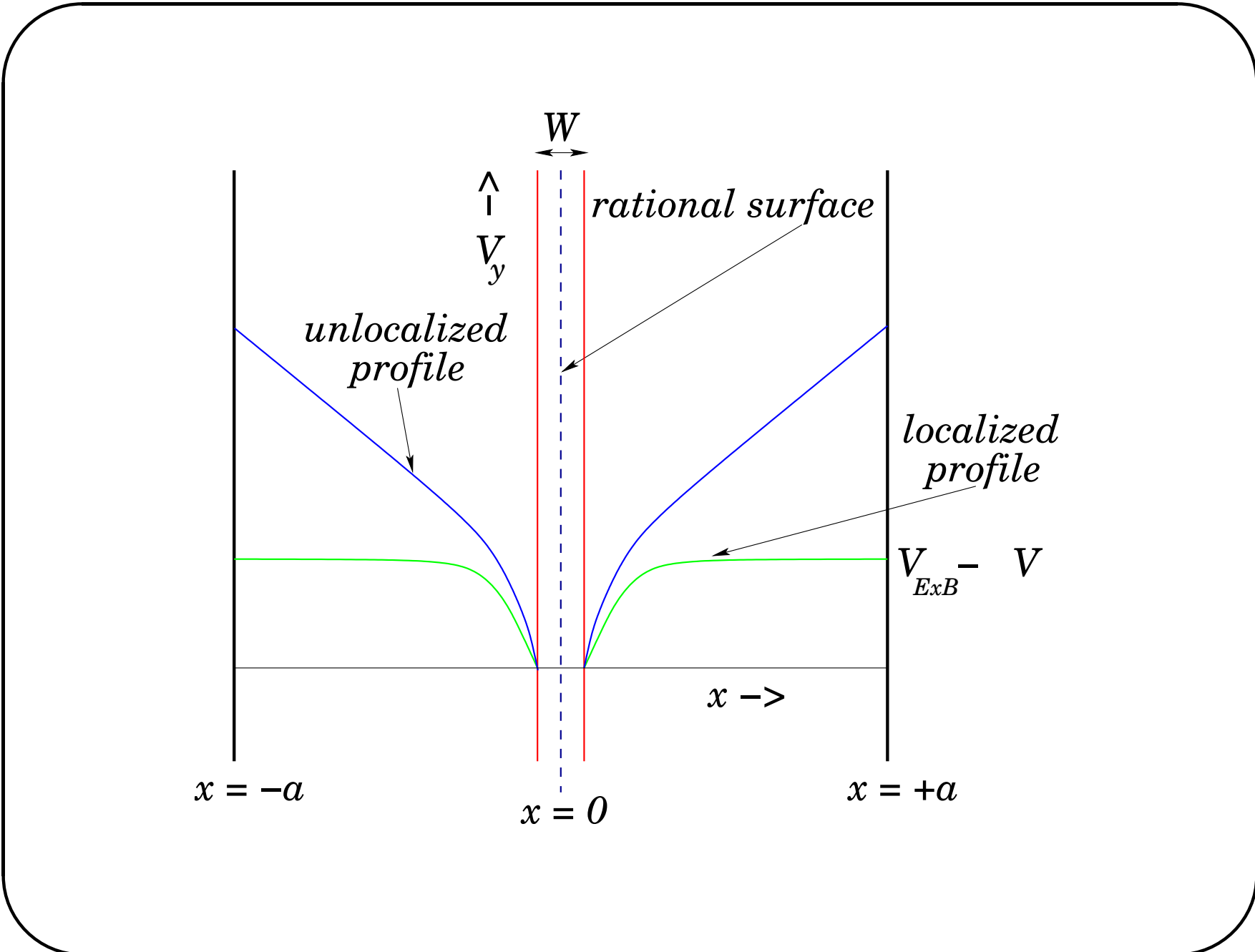
## MHD Flow - IV

- Note

$$V_y = \chi M \rightarrow |\chi| M_0$$

as  $|\chi|/W \rightarrow \infty$ .

- V-shaped velocity profile that extends over whole plasma.
- Expect *isolated* magnetic island to have *localized* velocity profile. Suggests that  $M_0 = 0$  for isolated island.
- Hence, zero MHD flow in island frame: *i.e.*, island propagates at local  $\vec{E} \times \vec{B}$  velocity.



## Rutherford Equation - I

- Asymptotic matching between inner and outer regions yields:

$$\Delta' \Psi = -4 \int_{+\Psi}^{-\infty} \langle J \cos \theta \rangle d\psi.$$

- In island frame, in absence of MHD flow, vorticity equation reduces to

$$[J, \psi] \simeq 0.$$

- Hence,

$$J = J(\psi).$$

## Rutherford Equation - II

- Ohm's law:

$$\frac{d\Psi}{dt} \cos \theta \simeq [\phi, \psi] + \eta J(\psi).$$

- Have shown there is no MHD-flow [*i.e.*,  $\phi \sim O(1)$ ], but can still be *resistive flow* [*i.e.*,  $\phi \sim O(\eta)$ ].
- Eliminate resistive flow by flux-surface averaging:

$$\frac{d\Psi}{dt} \langle \cos \theta \rangle \simeq \eta J(\psi) \langle 1 \rangle.$$

- Hence,

$$\Delta' \Psi \simeq -\frac{4}{\eta} \frac{d\Psi}{dt} \int_{+\Psi}^{-\infty} \frac{\langle \cos \theta \rangle^2}{\langle 1 \rangle} d\psi.$$

## Rutherford Equation - III

- Use  $W = 4 \sqrt{\Psi}$ , and evaluate integral. Obtain *Rutherford island width evolution equation*:<sup>a</sup>

$$\frac{0.823}{\eta} \frac{dW}{dt} \simeq \Delta'.$$

- According to Rutherford equation, island grows *algebraically* on *resistive time-scale*.
- Rutherford equation does not predict island saturation.

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<sup>a</sup>P.H. Rutherford, Phys. Fluids **16**, 1903 (1973).

## Rutherford Equation - IV

- Higher order asymptotic matching between inner and outer regions yields:<sup>a</sup>

$$\frac{0.823}{\eta} \frac{dW}{dt} \simeq \Delta' - 0.41 \left( -\frac{d^4 B_y^{(0)} / dx^4}{d^2 B_y^{(0)} / dx^2} \right)_{x=0} W.$$

- Hence, saturated ( $d/dt = 0$ ) island width is

$$W_0 = \frac{\Delta'}{0.41} \left( -\frac{d^2 B_y^{(0)} / dx^2}{d^4 B_y^{(0)} / dx^4} \right)_{x=0}.$$

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<sup>a</sup>F. Militello, and F. Porcelli, Phys. Plasmas **11**, L13 (2004). D.F. Escande, and M. Ottaviani, Physics Lett. A **323**, 278 (2004).

## Main Predictions of MHD Theory

- Tearing mode unstable if  $\Delta' > 0$ .
- Island propagates at local  $\vec{E} \times \vec{B}$  velocity at rational surface.
- Island grows algebraically on resistive time-scale.
- Saturated island width:

$$W_0 = \frac{\Delta'}{0.41} \left( -\frac{d^2 B_y^{(0)} / dx^2}{d^4 B_y^{(0)} / dx^4} \right)_{x=0} .$$

# 4: Profile Modification

## Temperature Flattening - I<sup>a</sup>

- For sake of simplicity, assume cold ions. Let  $T$  be electron temperature profile (minus uniform background value,  $T_e$ ).
- In immediate vicinity of island, unperturbed profile is

$$T(\mathbf{x}) \simeq T_e \frac{\chi}{L_T},$$

where  $L_T$  is equilibrium temperature gradient scale-length.

- Perturbed temperature profile determined by competition between parallel and perpendicular heat transport:

$$\chi_{\parallel} \nabla_{\parallel}^2 T + \chi_{\perp} \nabla_{\perp}^2 T \simeq 0.$$

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<sup>a</sup>R. Fitzpatrick, Phys. Plasmas **2**, 825 (1995).

## Temperature Flattening - II

- Parallel transport term attempts to make temperature a **flux-surface function**. Cannot have odd flux-surface function inside island separatrix. So, if  $T = T(\psi)$  then temperature is **flattened** inside separatrix.
- Perpendicular transport term attempts to relax temperature profile to unperturbed profile. Opposes temperature flattening.
- Temperature is flattened inside separatrix if parallel transport term dominates perpendicular transport term.

## Temperature Flattening - III

- Have

$$\nabla_{\parallel} \simeq \frac{k W}{L_s}.$$

- Also

$$\nabla_{\perp} \simeq \frac{1}{W}.$$

- Hence, parallel transport dominates perpendicular transport when

$$W \gg W_c,$$

where

$$k W_c \sim \left( \frac{\chi_{\perp}}{\chi_{\parallel}} \right)^{1/4} (k L_s)^{1/2}.$$

## Temperature Flattening - IV

- In “collisionless” fusion plasmas, Braginskii<sup>a</sup> expression for parallel heat flux due to conduction impossibly large: *i.e.*,

$$\chi_{\parallel} \nabla_{\parallel} T \gg n_e v_e T,$$

where  $n_e$  is uniform background electron number density, and  $v_e$  is thermal velocity.

- In this situation, parallel heat transport becomes *convective* in nature, rather than *diffusive*. Can model this effect by “flux limiting” heat flux: *i.e.*, replace  $\chi_{\parallel} \nabla_{\parallel} T$  by  $n_e v_e T$ .
- Leads to more realistic expression for critical island width<sup>b</sup>

$$k W_c \sim \left( \frac{\chi_{\perp}}{n_e v_e L_s} \right)^{1/3} (k L_s)^{2/3}.$$

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<sup>a</sup>S.I. Braginskii, Reviews of Plasma Physics, (Consultants Bureau, 1965)

<sup>b</sup>N.N. Gorolenkov, *et al.*, Phys. Plasmas **3**, 3379 (1996).

## Temperature Flattening - IV

- Temperature profile only flattened inside separatrix when island width exceeds critical value.
- Critical width much smaller than minor radius in conventional tokamak.
- Temperature flattening implies complete loss of radial energy confinement across island.

## Density Flattening - I <sup>a</sup>

- Let  $n$  be electron number density (minus uniform background value,  $n_e$ ).
- In immediate vicinity of magnetic island, unperturbed density profile is

$$n(x) \simeq n_e \frac{x}{L_n},$$

where  $L_n$  is equilibrium density gradient scale-length.

- Sound waves propagating along magnetic field-lines act to make density a **flux-surface function**. Cannot have odd flux-surface function inside island separatrix. So, if  $n = n(\psi)$  then density is **flattened** inside separatrix.

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<sup>a</sup>R. Fitzpatrick, Phys. Plasmas **2**, 825 (1995).

## Density Flattening - II

- Expect sound waves to flatten density profile when <sup>a</sup>

$$(\vec{V}_* \cdot \nabla) n \ll c_s \nabla_{\parallel} n,$$

where  $c_s = \sqrt{T_e/m_i}$  is (unnormalized) *sound speed*, and

$$\vec{V}_* = c_s \frac{\rho}{L_n} \vec{e}_y$$

is *diamagnetic velocity* due to equilibrium density gradient.

Furthermore,  $\rho = c_s/(e B_z/m_i)$  is ion Larmor radius calculated with electron temperature.

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<sup>a</sup>A.I. Smolyakov, Plasma Phys. Control. Fusion **35**, 657 (1993).

## Density Flattening - III

- Density profile flattened when island width exceeds critical value

$$W_c \sim \rho \frac{L_s}{L_n}.$$

- In typical tokamak plasma, critical width for density flattening generally considerably larger than that for temperature flattening, but still much smaller than minor radius.
- If island width exceeds critical value for density flattening then pressure profile completely flattened inside island separatrix. Implies complete loss of radial energy and particle confinement across island.

# 5: Bootstrap Current Destabilization

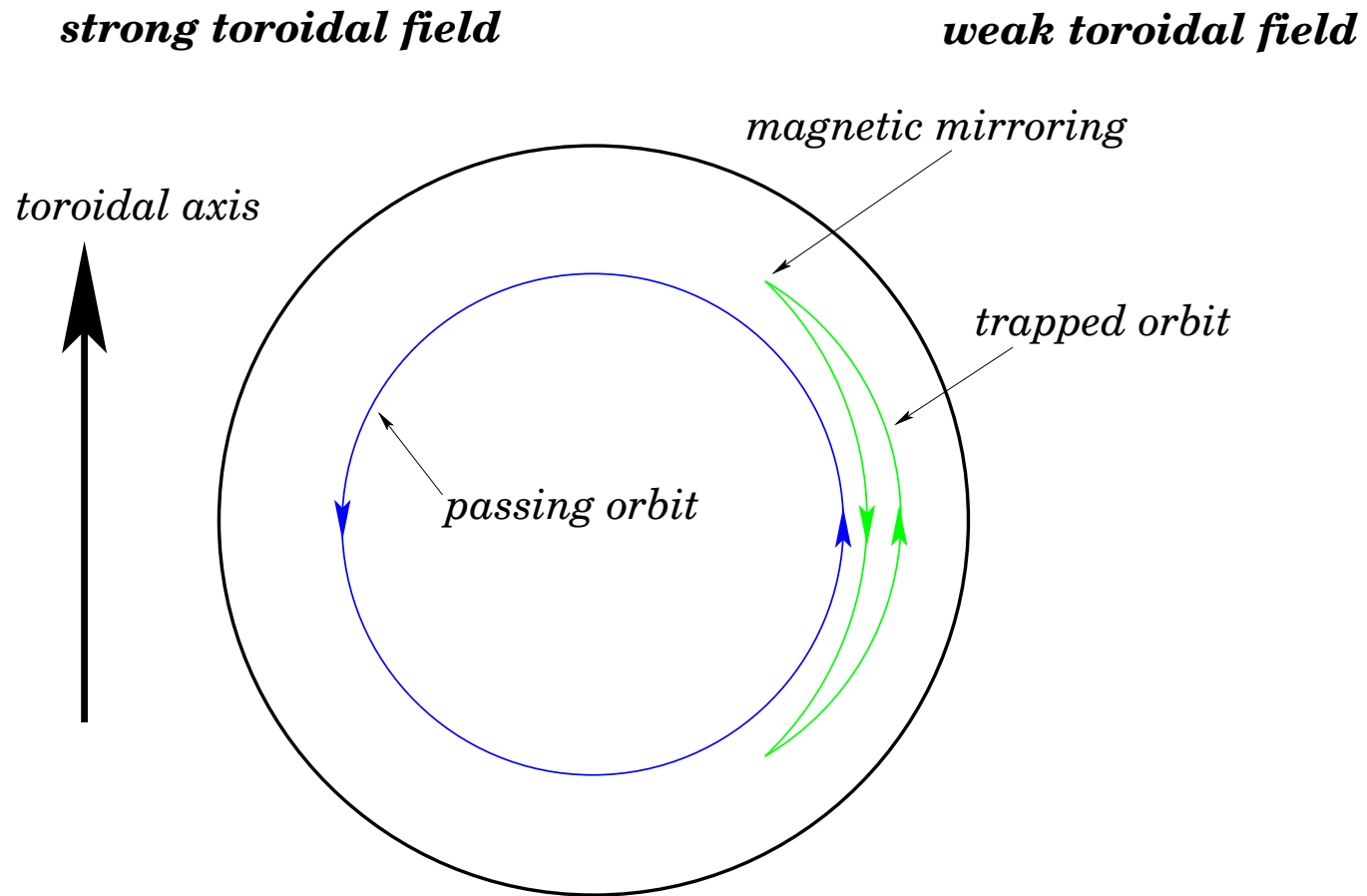
## Neoclassical Effects

- So-called *neoclassical effects*<sup>a</sup> in tokamak plasmas arise from essential *toroidicity* of such plasmas, combined with **extremely long mean-free-path** of electrons and ions streaming along magnetic field-lines, due to high plasma temperature.

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<sup>a</sup>*The Theory of Toroidally Confined Plasmas*, 2nd Rev. Edition, R.B. White (World Scientific, 2006).

# Trapped and Passing Particles



## Bootstrap Current - I

- In toroidal plasma, friction between trapped and passing electrons leads to appearance of non-inductive *bootstrap current* in Ohm's law:<sup>a</sup>

$$\frac{d\Psi}{dt} \cos \theta \simeq [\phi, \psi] + \eta [J(\psi) - J_{\text{boot}}],$$

where

$$J_{\text{boot}} = -1.46 \sqrt{\epsilon} B_{\theta}^{-1} \frac{dP}{dx}.$$

Here,  $\epsilon$  is inverse aspect-ratio,  $1.46 \sqrt{\epsilon}$  is measure of fraction of trapped-particles,  $B_{\theta}$  is poloidal magnetic field-strength, and  $P$  is plasma pressure.

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<sup>a</sup>M.N. Rosenbluth, R.D. Hazeltine, and F.L. Hinton, Phys. Fluids **15**, 116 (1972).

## Bootstrap Current - II

- For sufficiently wide island, pressure profile *flattened* inside separatrix.
- Bootstrap current consequently disappears inside separatrix.
- Absence of bootstrap current inside separatrix, and continued presence outside, leads to *destabilizing* term in Rutherford island equation:<sup>a</sup>

$$\frac{0.823}{\eta} \frac{dW}{dt} \simeq \Delta' + 9.25 \sqrt{\epsilon} \beta \frac{L_s}{L_p} \frac{B_z}{B_\theta} \frac{1}{W},$$

where  $\beta = \mu_0 n_e T_e / B_z^2$ , and  $L_p^{-1} = L_n^{-1} + L_T^{-1}$ .

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<sup>a</sup>R. Carrera, R.D. Hazeltine, and M. Kotschenreuther, Phys. Fluids **29**, 899 (1986).

## Neoclassical Tearing Modes

- A *neoclassical tearing mode* (NTM) is an *intrinsically stable* ( $\Delta' < 0$ ) tearing mode destabilized by bootstrap term.
- Bootstrap term in Rutherford equation relatively large, especially at small island widths. Would expect plasma to be filled with NTMs, and confinement to be wrecked.<sup>a</sup>
- This is not observed to be case. Experimental evidence for *threshold island width* above which NTMs grow, but below which they decay.<sup>b</sup>
- Suggests presence of *stabilizing effect* in Rutherford equation that opposes destabilizing bootstrap term.

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<sup>a</sup>C.C. Hegna, J.D. Callen, Phys. Fluids B **4**, 1855 (1992).

<sup>b</sup>O. Sauter, *et al.*, Phys. Plasmas **4**, 1654 (1997).

## Incomplete Pressure Flattening - I

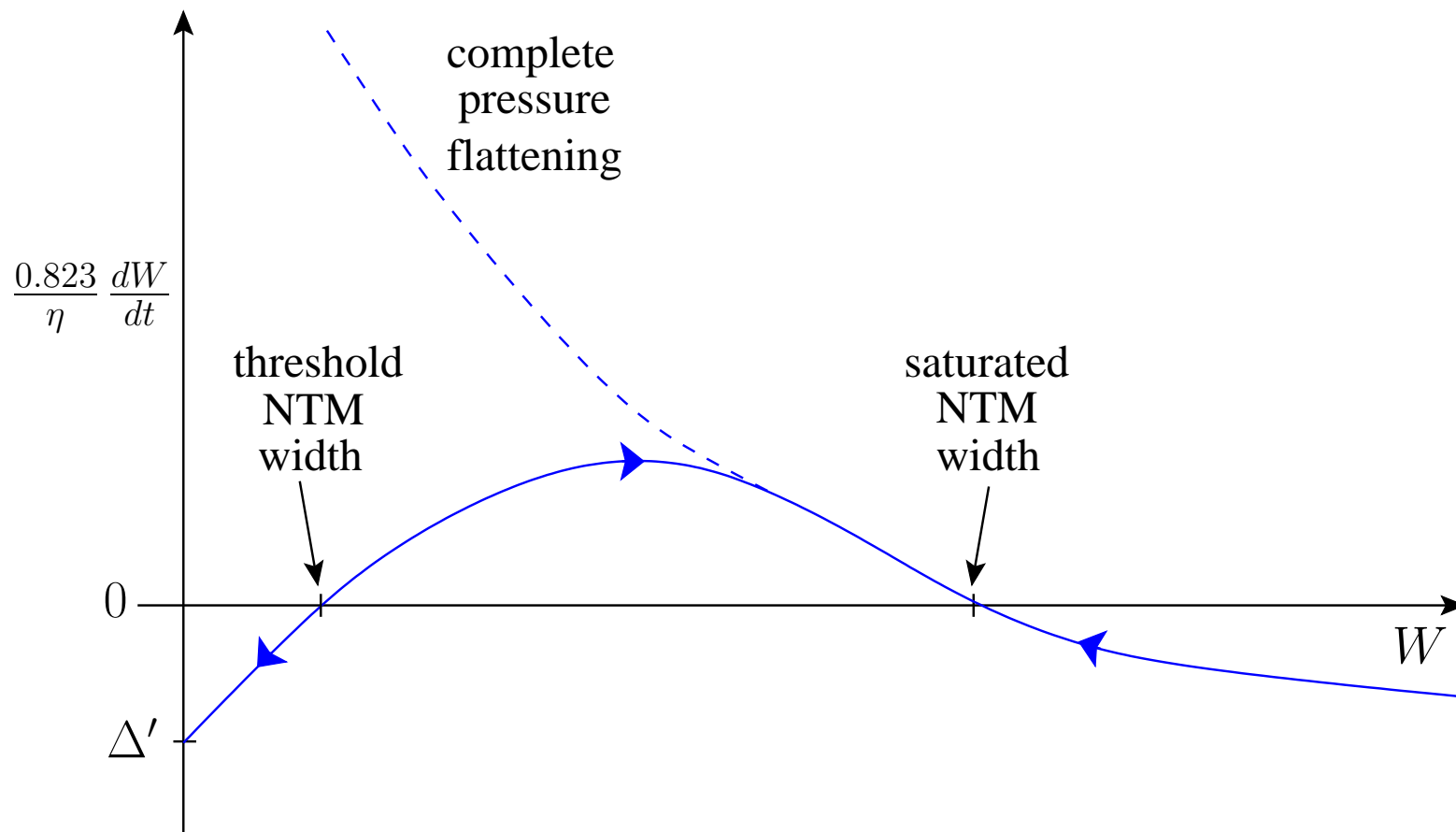
- Bootstrap destabilization is caused by flattening of pressure profile inside island separatrix. If there is no flattening then there is no destabilization.
- Pressure flattening only occurs when island width exceeds critical value  $W_c$ .
- When incomplete pressure flattening incorporated into Rutherford equation find that <sup>a</sup>

$$\frac{0.823}{\eta} \frac{dW}{dt} \simeq \Delta' + 9.25 \sqrt{\epsilon} \beta \frac{L_s}{L_p} \frac{B_z}{B_\theta} \frac{W}{W^2 + W_c^2}.$$

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<sup>a</sup>R. Fitzpatrick, Phys. Plasmas **2**, 825 (1995).

## Incomplete Pressure Flattening - II



# 6: Drift-MHD Theory

## Introduction

- In drift-MHD model, which is far more accurate representation of tokamak plasma than MHD model, analysis retains *charged particle drift velocities*, in addition to  $\vec{E} \times \vec{B}$  velocity.
- Essentially *two-fluid* theory of plasma.
- Characteristic length-scale,  $\rho$ , is *ion Larmor radius calculated with electron temperature*.
- Characteristic velocity is *diamagnetic velocity*,  $V_*$ .
- Normalize all lengths to  $\rho$ , and all velocities to  $V_*$ .

## Basic Assumptions

- Retain slab model, for sake of simplicity.
- Assume island sufficiently wide that  $T = T(\psi)$ .
- Assume  $T_i/T_e = \tau = \text{constant}$ , for sake of simplicity, where  $T_i$  and  $T_e$  are complete electron and ion temperature profiles.

## Basic Definitions

- Variables:
  - $\psi$  - magnetic flux-function.
  - $J$  - parallel current density.
  - $\phi$  - guiding-center (*i.e.*, MHD) stream-function.
  - $\mathcal{U}$  - parallel ion vorticity.
  - $n$  - electron number density (minus uniform background).
  - $V_z$  - parallel ion velocity.
- Parameters:
  - $\alpha = (L_n/L_s)^2$ , where  $L_s$  is magnetic shear length, and  $L_n$  is density gradient scale-length.
  - $\eta$  - resistivity.  $D$  - (perpendicular) particle diffusivity.  $\mu_{i/e}$  - (perpendicular) ion/electron viscosity.

## Drift-MHD Equations - I

- Steady-state drift-MHD equations: <sup>a</sup>

$$\psi = -x^2/2 + \Psi \cos \theta, \quad \mathbf{U} = \nabla^2 \phi,$$

$$0 = [\phi - n, \psi] + \eta J,$$

$$0 = [\phi, \mathbf{U}] - \frac{\tau}{2} \{ \nabla^2 [\phi, n] + [\mathbf{U}, n] + [\nabla^2 n, \phi] \} \\ + [J, \psi] + \mu_i \nabla^4 (\phi + \tau n) + \mu_e \nabla^4 (\phi - n),$$

$$0 = [\phi, n] + [V_z + J, \psi] + D \nabla^2 n,$$

$$0 = [\phi, V_z] + \alpha [n, \psi] + \mu_i \nabla^2 V_z.$$

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<sup>a</sup>R.D. Hazeltine, M. Kotschenreuther, and P.J. Morrison, Phys. Fluids **28**, 2466 (1985).

## Drift-MHD Equations - II

- Symmetry:  $\psi, J, V_z$  even in  $x$ .  $\phi, n, U$  odd in  $x$ .
- Boundary conditions as  $|x|/W \rightarrow \infty$ :
  - $n \rightarrow -(1 + \tau)^{-1} x$ .
  - $\phi \rightarrow -V x$ .
  - $J, U, V_z \rightarrow 0$ .
- Here,  $V$  is island phase-velocity in  $\vec{E} \times \vec{B}$  frame.
- $V = 1$  corresponds to island propagating with electron fluid.  
 $V = -\tau$  corresponds to island propagating with ion fluid.
- Expect

$$1 \gg \alpha \gg \eta, D, \mu_i, \mu_e.$$

## Electron Fluid

- Ohm's law:

$$0 = [\phi - n, \psi] + \eta J.$$

- Since  $\eta \ll 1$ , first term potentially much larger than second.
- To lowest order:

$$[\phi - n, \psi] \simeq 0.$$

- Follows that

$$n = \phi + H(\psi) :$$

*i.e.*, electron stream-function  $\phi_e = \phi - n$  is *flux-surface function*.  
Electron fluid flow constrained to be around flux-surfaces.

## Sound Waves

- Parallel flow equation:

$$0 = [\phi, V_z] + \alpha [n, \psi] + \mu_i \nabla^2 V_z.$$

- Highlighted term dominant provided

$$W \gg \alpha^{-1/2} = L_s/L_n.$$

- If this is case then, to lowest order,

$$n = n(\psi),$$

which implies  $n = 0$  inside separatrix.

- So, if island sufficiently wide then *sound-waves* able to *flatten density profile* inside island separatrix.

## Subsonic vs. Supersonic Islands

- Wide islands satisfying

$$W \gg L_s/L_n$$

termed *subsonic* islands. Expect such islands to exhibit flattened density profile within separatrix. Subsonic islands strongly coupled to both electron and ion fluids.

- Narrow islands satisfying

$$W \ll L_s/L_n$$

termed *supersonic* islands. No flattening of density profile within separatrix. Supersonic islands strongly coupled to electron fluid, but only weakly coupled to ion fluid.

# 7: Subsonic Island Theory

## Introduction<sup>a</sup>

- To lowest order:

$$\phi = \phi(\psi), \quad n = n(\psi).$$

- Follows that both electron stream-function,  $\phi_e = \phi - n$ , and ion stream-function,  $\phi_i = \phi + \tau n$ , are flux-surface functions. Both electron and ion fluid flow constrained to follow flux-surfaces.
- Let

$$M(\psi) = d\phi/d\psi, \quad L(\psi) = dn/d\psi.$$

- Follows that

$$V_{E \times B y} = x M, \quad V_{e y} = x (M - L), \quad V_{i y} = x (M + \tau L).$$

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<sup>a</sup>R. Fitzpatrick, F.L. Waelbroeck, Phys. Plasmas **12**, 022307 (2005).

## Density Flattening

- By symmetry, both  $M(\psi)$  and  $L(\psi)$  are *odd* functions of  $x$ .  
Hence,

$$M(\psi) = L(\psi) = 0$$

inside separatrix: *i.e.*, no electron/ion flow within separatrix in island frame.

- Electron/ion fluids constrained to propagate with island inside separatrix.
- Density profile *flattened* within separatrix.

## Analysis - I

- Density equation reduces to

$$0 \simeq [V_z + J, \psi] + D \nabla^2 n.$$

- Vorticity equation reduces to

$$0 \simeq [-M U - (\tau/2)(L U + M \nabla^2 n) + J, \psi] \\ + \mu_i \nabla^4 (\phi + \tau n) + \mu_e \nabla^4 (\phi - n).$$

- Flux-surface average both equations, recalling that  $\langle [A, \psi] \rangle = 0$ .

## Analysis - II

- Obtain

$$\langle \nabla^2 \mathbf{n} \rangle \simeq 0,$$

and

$$(\mu_i + \mu_e) \langle \nabla^4 \Phi \rangle + (\mu_i \tau - \mu_e) \langle \nabla^4 \mathbf{n} \rangle \simeq 0.$$

- Solution outside separatrix:

$$M(\psi) = -\frac{(\mu_i \tau - \mu_e)}{(\mu_i + \mu_e)} L(\psi) + F(\psi),$$

where

$$L(\psi) = -\text{sgn}(x) L_0 / \langle x^2 \rangle,$$

and  $F(\psi)$  is previously obtained MHD profile:

$$F(\psi) = \text{sgn}(x) F_0 \int_{-\Psi}^{\psi} d\psi / \langle x^4 \rangle \bigg/ \int_{-\Psi}^{-\infty} d\psi / \langle x^4 \rangle.$$

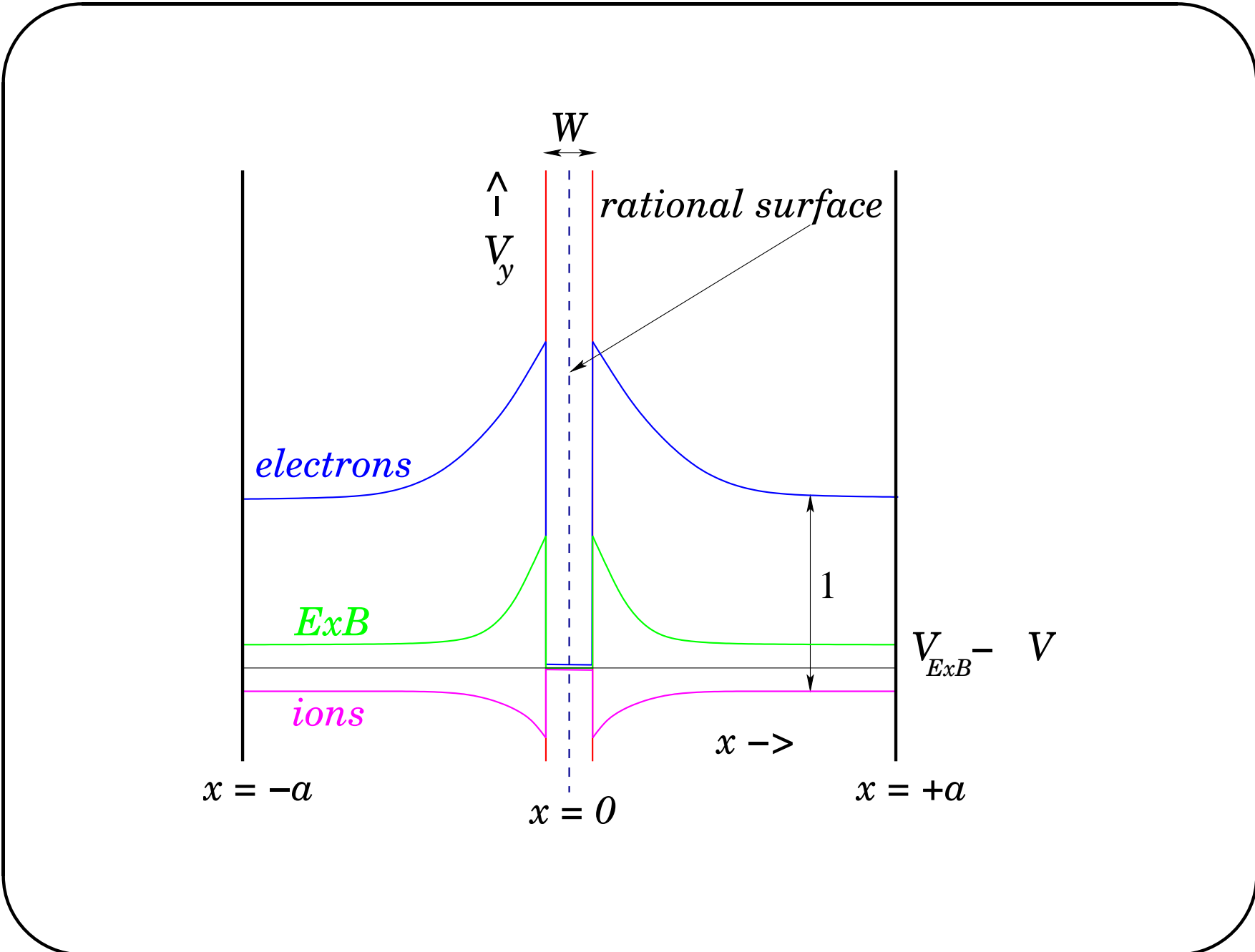
## Velocity Profiles

- As  $|x|/W \rightarrow \infty$  then  $x L \rightarrow L_0$  and  $x F \rightarrow |x| F_0$ .
- $L(\psi)$  corresponds to *localized* velocity profile.  $F(\psi)$  corresponds to *non-localized* profile. Require localized profile, so  $F_0 = 0$ .
- Velocity profiles outside separatrix (using b.c. on  $n$ ):

$$V_{y i} \simeq + \frac{\mu_e}{\mu_i + \mu_e} \frac{|x|}{\langle x^2 \rangle},$$

$$V_{y E \times B} \simeq - \frac{(\mu_i \tau - \mu_e)}{(1 + \tau)(\mu_i + \mu_e)} \frac{|x|}{\langle x^2 \rangle},$$

$$V_{y e} = - \frac{\mu_i}{\mu_i + \mu_e} \frac{|x|}{\langle x^2 \rangle}.$$



## Island Propagation

- As  $|\chi|/W \rightarrow \infty$  expect  $V_{y \text{ E} \times \text{B}} \rightarrow V_{\text{EB}} - V$ , where  $V_{\text{EB}}$  is unperturbed (*i.e.*, no island)  $\vec{E} \times \vec{B}$  velocity at rational surface (in lab. frame), and  $V$  is island phase-velocity (in lab. frame).

- Hence

$$V = V_{\text{EB}} + \frac{(\mu_i \tau - \mu_e)}{(1 + \tau)(\mu_i + \mu_e)}.$$

- But unperturbed ion/electron fluid velocities (in lab. frame):

$$V_i = V_{\text{EB}} + \tau/(1 + \tau), \quad V_e = V_{\text{EB}} - 1/(1 + \tau).$$

- Hence

$$V = \frac{\mu_i}{\mu_i + \mu_e} V_i + \frac{\mu_e}{\mu_i + \mu_e} V_e.$$

So, island phase-velocity is *viscosity weighted average* of unperturbed ion/electron fluid velocities.

## Polarization Term - I

- Vorticity equation yields

$$J_c \simeq \frac{1}{2} \left( x^2 - \frac{\langle x^2 \rangle}{\langle 1 \rangle} \right) \frac{d[M (M + \tau L)]}{d\psi} + I(\psi)$$

outside separatrix, where  $J_c$  is part of  $J$  with  $\cos \theta$  symmetry.

- As before, flux-surface average of Ohm's law yields:

$$\langle J_c \rangle = I(\psi) \langle 1 \rangle = \eta^{-1} \frac{d\Psi}{dt} \langle \cos \theta \rangle.$$

- Hence

$$J_c \simeq \frac{1}{2} \left( x^2 - \frac{\langle x^2 \rangle}{\langle 1 \rangle} \right) \frac{d[M (M + \tau L)]}{d\psi} + \eta^{-1} \frac{d\Psi}{dt} \frac{\langle \cos \theta \rangle}{\langle 1 \rangle}.$$

## Polarization Term - II

- Asymptotic matching between inner and outer regions yields:

$$\Delta' \Psi = -4 \int_{+\Psi}^{-\infty} \langle J_c \cos \theta \rangle d\psi.$$

- Evaluating flux-surface integrals, making use of previous solutions for  $M$  and  $L$ , obtain modified Rutherford equation:

$$\frac{0.823}{\eta} \frac{dW}{dt} \simeq \Delta' + 5.52 \beta (V - V_{EB}) (V - V_i) \frac{L_s^2}{L_n^2} \frac{1}{W^3}.$$

- New term is due to *polarization current* associated with ion fluid flow around curved island flux-surfaces (in island frame).  
Obviously, new term is zero if island propagates with ion fluid:  
*i.e.*,  $V = V_i$ .

## Main Predictions of Subsonic Island Theory

- Results limited to large islands: *i.e.*, large enough for sound waves to flatten density profile.
- Island propagates at (perpendicular) viscosity weighted average of unperturbed (no island) ion and electron fluid velocities.
- Polarization term in Rutherford equation is **stabilizing** provided ion (perpendicular) viscosity greatly exceeds electron (perpendicular) viscosity (which is what we expect), and destabilizing otherwise.
- Polarization term ( $\propto W^{-3}$ ) dominates bootstrap term ( $\propto W^{-1}$ ) at small island widths, and *vice versa* at large island widths. Thus, polarization term can also provide threshold effect that prevents NTMs from growing until they exceed critical island width.

# 8: Supersonic Island Theory

## Drift-MHD Equations<sup>a</sup>

- Steady-state drift-MHD equations (with  $\tau = 0$ , since ion diamagnetic effects largely irrelevant to supersonic islands):

$$\psi = -x^2/2 + \Psi \cos \theta, \quad \mathcal{U} = \nabla^2 \phi,$$

$$0 = [\phi - n, \psi] + \eta J,$$

$$0 = [\phi, \mathcal{U}] + [J, \psi] + \mu_i \nabla^4 \phi,$$

$$0 = [\phi, n] + [V_z + J, \psi] + D \nabla^2 n,$$

$$0 = [\phi, V_z] + \alpha [n, \psi] + \mu_i \nabla^2 V_z.$$

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<sup>a</sup>R. Fitzpatrick, and F.L. Waelbroeck, Phys. Plasmas **14**, 122502 (2007).

## Zero- $\alpha$ Solution

- By definition, highlighted term small for supersonic islands.
- If term completely neglected, obtain trivial solution:

$$\phi = n = -x, \quad U = V_z = J = 0.$$

- Island propagates with *electron fluid*.
- Island does not perturb ion fluid, so *zero polarization current*.

## Small- $\alpha$ Solution

- Assume that highlighted term small, but not negligible. Perturb about zero- $\alpha$  solution.
- So

$$\phi = -x + \delta\phi, \quad n = -x + \delta n,$$

where  $\delta\phi, \delta n, U, V_z, J$  all  $O(\alpha) \ll 1$ .

## Analysis - I

- Lowest order solution:

$$\delta n = \delta\phi + H(\psi),$$

$$J = -\tilde{G} + (\alpha/2) \tilde{x}^2,$$

$$V_z = -\alpha (W/4)^2 \cos \theta,$$

where  $\tilde{A} \equiv A - \langle A \rangle / \langle 1 \rangle$ .

- Here,  $G = -x H'$ . Now,  $G = 0$  inside separatrix, but outside separatrix

$$G = |x| \left( \frac{\langle x v \rangle + \alpha (W/4)^4}{\langle x^2 \rangle} \right),$$

where  $v = -\delta\phi_x$ .

## Analysis - II

- Perturbed velocity  $v$  satisfies

$$v_{xx} = (D/\mu) (\bar{v} - \bar{G}) - (G - \bar{G}) - \alpha (W/4)^2 \cos \theta,$$

where  $\bar{\dots}$  denotes a  $\theta$ -average at constant  $x$ .

- Boundary conditions:  $v_x = 0$  at  $x = 0$ , and

$$v \rightarrow v_i + v'_i |x| - (\alpha/2) (W/4)^2 x^2 \cos \theta$$

as  $|x| \rightarrow \infty$ .

- Above equation highly nonlinear, but can be solved via iteration.

## Need for Intermediate Layer

- Inner region island solution does not satisfy  $J \rightarrow 0$  as  $|x| \rightarrow \infty$ : *i.e.*, it does not asymptote to ideal-MHD solution in outer region.
- Require *intermediate layer* between island and outer region to allow proper matching.
- Intermediate layer much wider than island, so governed by *linear* physics.

## Intermediate Layer - I

- Write

$$\phi(x, \theta) = -x + \overline{\delta\phi}(x) + \check{\phi}(x) e^{i\theta}.$$

- Neglect all transport terms except ion viscosity.
- Linearized drift-MHD equations yield

$$\begin{aligned} \check{\phi}_{xx} &= \bar{v}_{xx} \check{\phi} - \left( \bar{v} - \frac{\alpha x^2}{1 - i\mu_i \alpha x^2} \right) \check{\phi} \\ &= - \left( \bar{v} - \frac{\alpha x^2}{1 - i\mu_i \alpha x^2} \right) \frac{(W/4)^2}{x}, \end{aligned}$$

where  $\bar{v} = -\overline{\delta\phi}_x$ .

## Intermediate Layer - II

- Mean velocity profile determined by *quasi-linear force balance*:

$$\bar{v}_{xx} = \frac{1}{2} \frac{\alpha^2 x^2}{1 + (\mu_i \alpha x^2)^2} |(W/4)^2 - x \check{\phi}|^2.$$

- Perturbed current:

$$\check{J} = (\check{\phi}_{xx} - \bar{v}_{xx} \check{\phi})/x.$$

## Intermediate Layer - III

- Boundary conditions as  $x \rightarrow 0$ :

$$\check{\phi} \rightarrow 0,$$

$$\bar{v} \rightarrow v_i + v'_i |x|.$$

- Boundary conditions as  $|x| \rightarrow \infty$ :

$$\check{\phi} \rightarrow (W/4)^2/x,$$

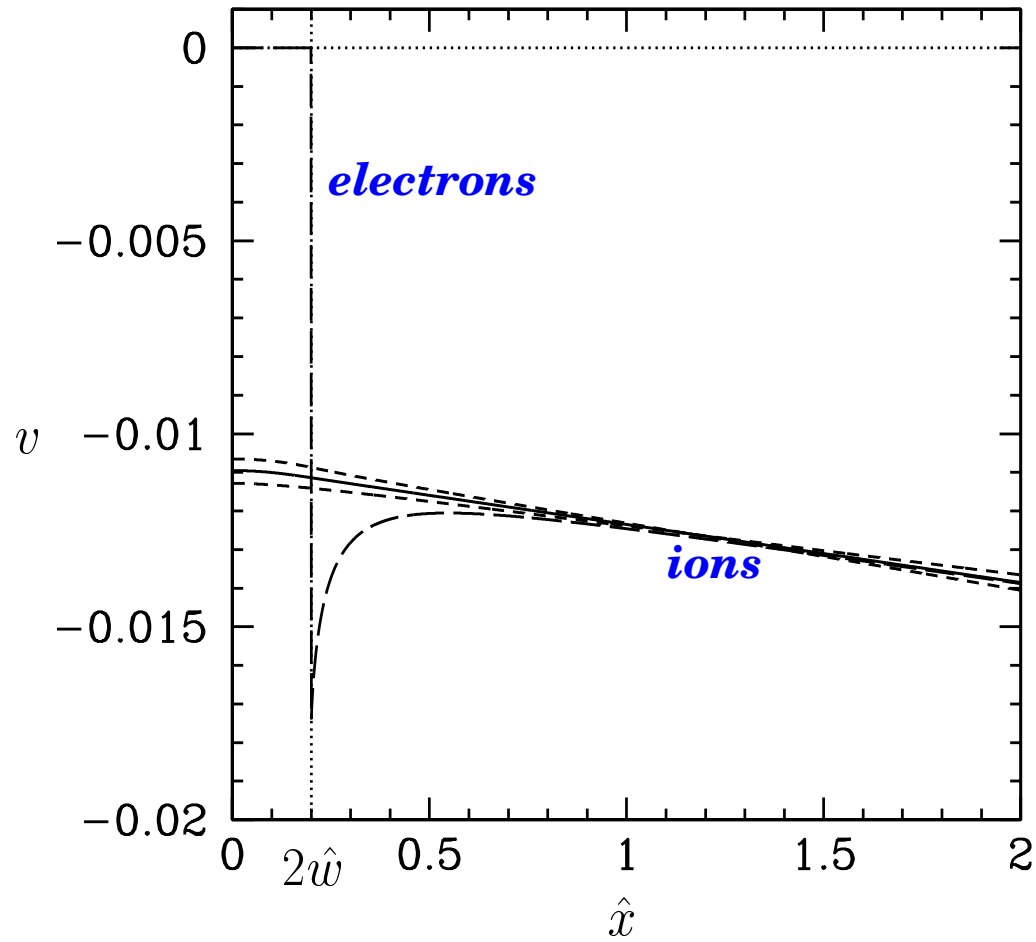
$$\bar{v} \rightarrow v_\infty + v'_\infty |x|.$$

- Large- $|x|$  boundary conditions ensure that  $\check{j} \rightarrow 0$ . So solution matches to ideal-MHD solution.

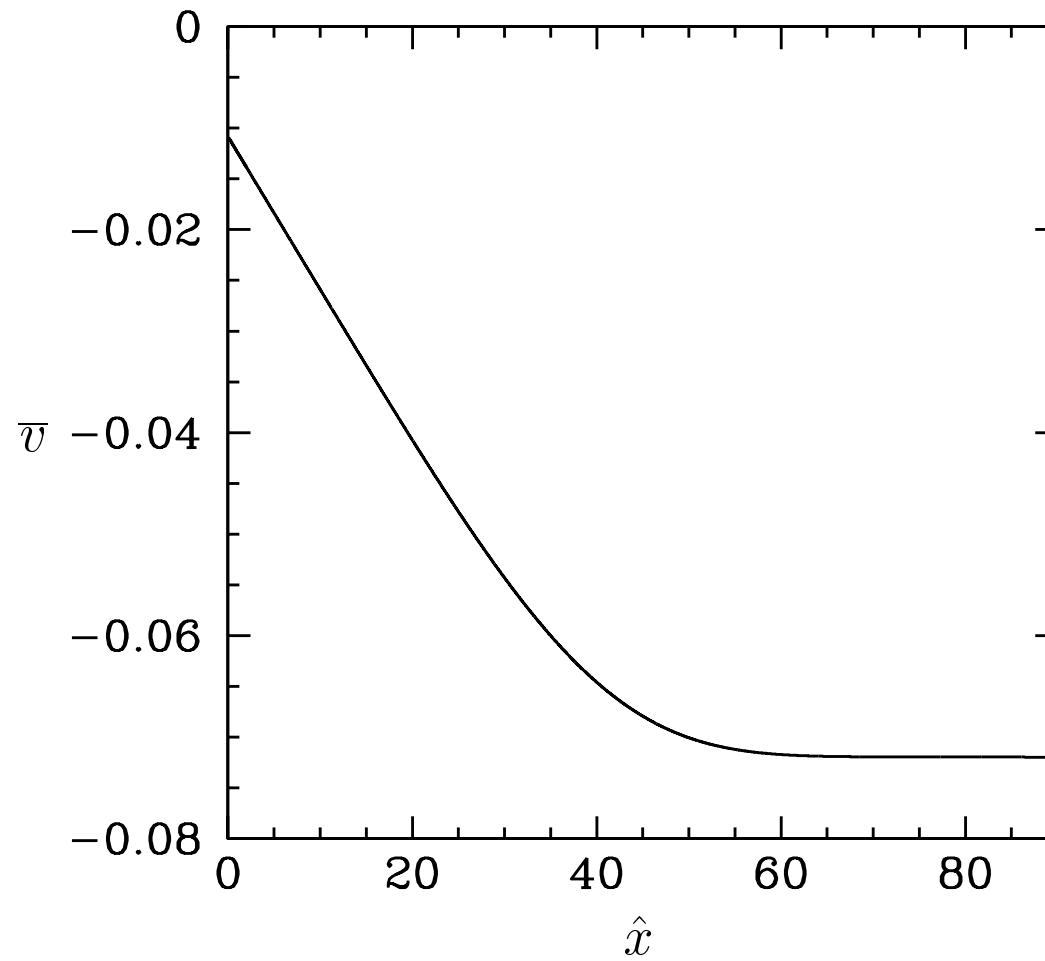
## Physics of Intermediate Layer

- Island launches *drift-acoustic waves* into intermediate layer.
- Waves are *absorbed* in layer (due to ion viscosity).
- Waves carry *momentum*.
- Momentum exchange between island and intermediate layer ensures that velocity gradient,  $v'_i$ , at inner boundary of layer not same as gradient,  $v'_\infty$ , at outer boundary.
- For isolated island solution, require  $v'_\infty = 0$ . This boundary condition *uniquely specifies* solution for given values of  $\alpha$ ,  $\mu_i$ ,  $D$ , etc.

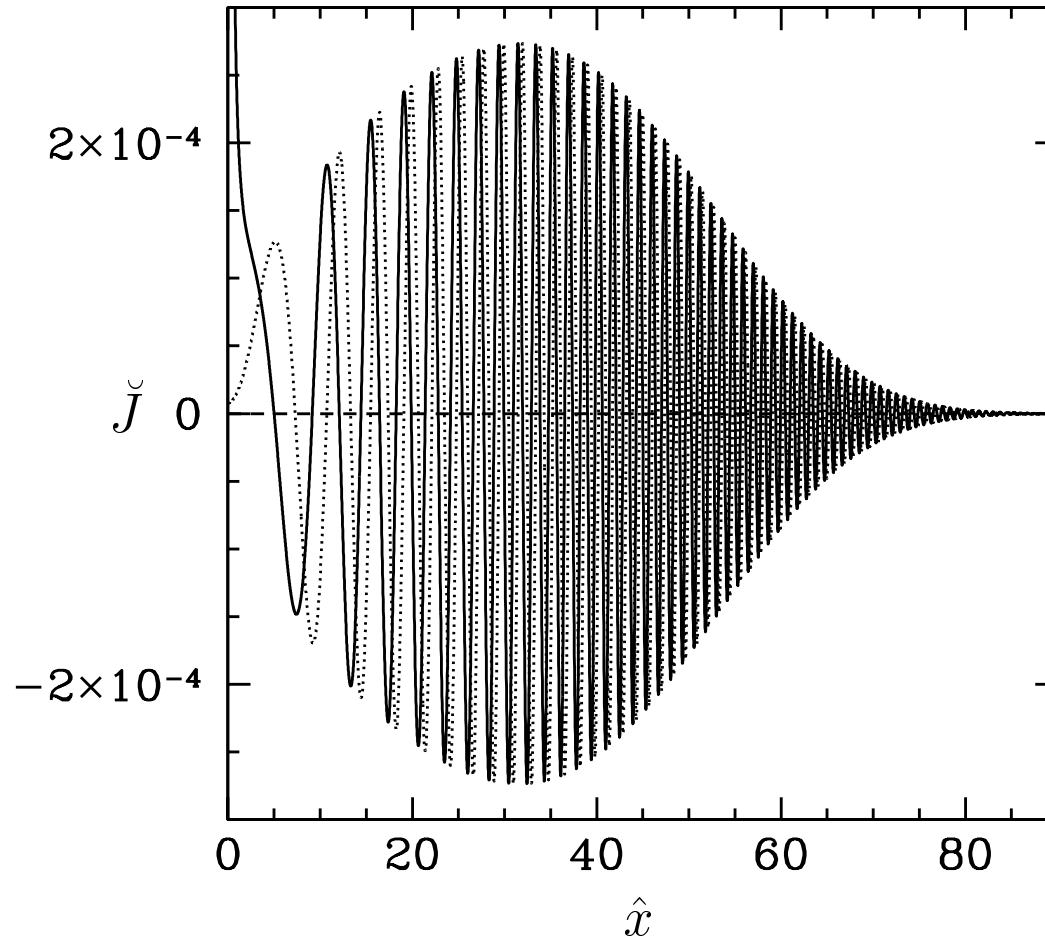
## Velocity in Island Region



## Velocity in Intermediate Layer



## Current in Intermediate Layer



## Island Propagation

- Island propagation velocity:

$$V = V_e - 0.27 (W/4)^3 \alpha^{3/4} D^{-1} - 0.24 (W/4)^4 \alpha^{1/3} \mu^{-4/3}.$$

- Island phase velocity close to unperturbed electron fluid velocity, but dragged slightly in ion direction due to sound-wave effects.

## Ion Polarization Term

- Rutherford Equation:

$$\frac{0.823}{\eta} \frac{dW}{dt} = \Delta' - \frac{\beta}{\alpha^{1/4}} \frac{L_s^2}{L_n^2} [1.5 + 0.38 (W/4)^2 D^{-1}] .$$

- Sound-wave effects ensure ion fluid slightly perturbed by island, generating polarization term in Rutherford equation. Term is *stabilizing*.

## Maximum Island Width

- Supersonic branch of solutions ceases to exist beyond *maximum island width*:

$$W_{\max} = 0.36 \alpha^{-1/12} D^{1/3}.$$

- Hypothesized that island bifurcates to subsonic solution branch when  $W > W_{\max}$ . This type of behavior has been observed in computer simulations.<sup>a</sup>

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<sup>a</sup>M. Ottaviani, F. Porcelli, and D. Grasso, Phys. Rev. Lett. **93**, 075001 (2004).

## Main Predictions of Supersonic Island Theory

- Results limited to small islands: *i.e.*, small enough that sound waves cannot flatten density profile.
- Islands phase velocities close to unperturbed electron fluid velocity, but dragged slightly in ion direction by sound wave effects.
- Islands radiate drift-acoustic waves.
- Momentum carried by drift-acoustic waves gives rise to strong velocity shear in region surrounding islands.
- Polarization term in Rutherford island equation is stabilizing.
- Supersonic branch ceases to exist above critical island width.

# 9: Other Effects

## Neoclassical Flow Damping

- Poloidal (and, sometimes, toroidal) flow strongly damped in low-collisionality plasmas typically found in tokamaks.
- Flow damping affects island propagation velocity, which modifies ion polarization term in Rutherford equation.
- Require drift-MHD island theory that takes flow damping into account.<sup>a</sup>

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<sup>a</sup>R. Fitzpatrick, and F.L. Waelbroeck, Phys. Plasmas **16**, 072507 (2009).

## Finite Trapped Ion Orbit Width

- Width of trapped ion orbit of order

$$\rho_{\theta} = (B_z/B_{\theta}) \rho.$$

- In conventional tokamak, trapped ion orbit width often comparable with island width.
- Kinetic analysis required to take finite orbit widths into account.<sup>a</sup>

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<sup>a</sup>A. Bergmann, E. Poli, and A.G. Peeters, Phys. Plasmas **12**, 072501 (2005).

## Magnetic Field-Line Curvature

- Magnetic field-line curvature in tokamak plasmas gives rise to particle drifts that are three-dimensional in nature, and cannot be captured in two-dimensional slab model.
- Three-dimensional island theory required to take curvature drifts into account.<sup>a</sup>

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<sup>a</sup>M. Kotschenreuther, R.D. Hazeltine, and P.J. Morrison, Phys. Fluids **28**, 294 (1985).

## Drift-Wave Turbulence

- Perpendicular transport that determines island profiles actually due to drift-wave turbulence.
- Radial extent of drift-wave eddies of order  $\rho$ . Hence, eddies can easily be comparable in width to island.
- Island theory in which island immersed in bath of drift-wave turbulence required when eddy width comparable with island width. Turbulence affects island by modifying island profiles. Island profiles affect drift-wave stability, and hence turbulence levels. Theory must self-consistently determine effect of turbulence on island, and effect of island on turbulence.<sup>a</sup>

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<sup>a</sup>F. Militello, F.L. Waelbroeck, R. Fitzpatrick, and W. Horton, Plasma Phys. Control. Fusion **51**, 015015 (2009).