

# Fundamentals of Magnetic Island Theory in Tokamaks

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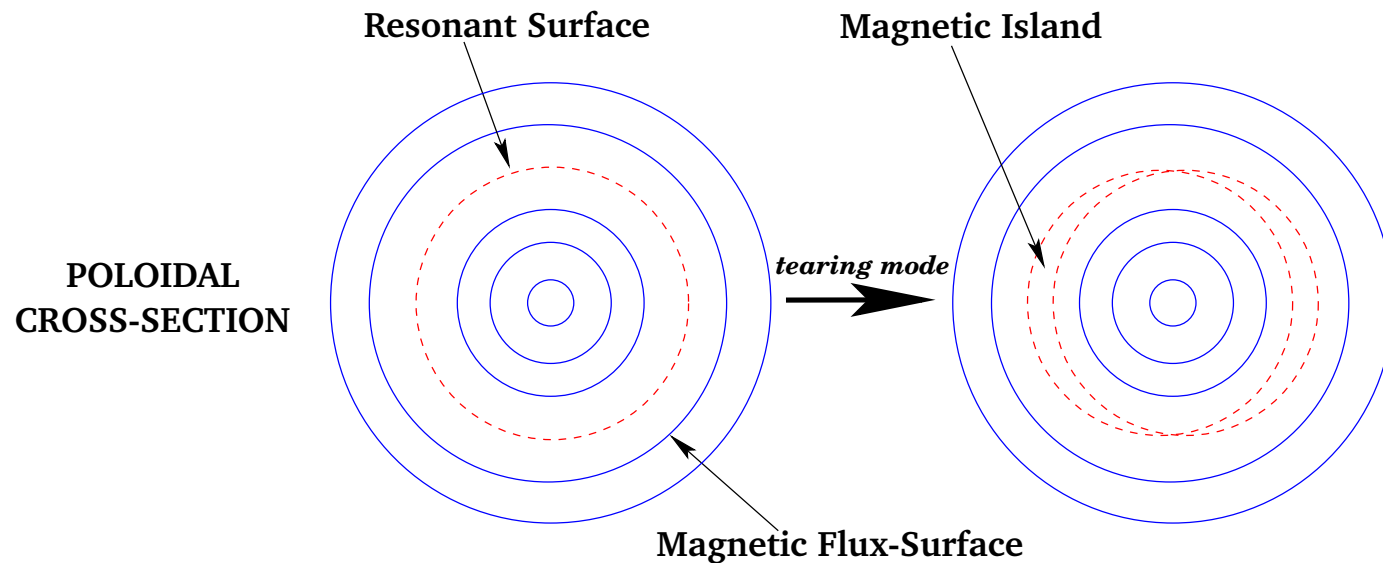
## Macroscopic Instabilities

- Two main types of macroscopic instabilities in tokamaks: <sup>a</sup>
  - Catastrophic “ideal” (*i.e.*, non-reconnecting) instabilities, which disrupt plasma in few micro-seconds. Can be avoided by limiting plasma pressure and current.
  - Slowly growing “tearing” instabilities, which reconnect magnetic flux-surfaces to form *magnetic islands*, thereby degrading their confinement properties. Much harder to avoid.

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<sup>a</sup>*MHD Instabilities*, G. Bateman (MIT, 1978).

# Magnetic Islands



- Helical structures, centered on *rational magnetic flux-surfaces* which satisfy  $\vec{k} \cdot \vec{B} = 0$ , where  $\vec{k}$  is wavenumber of mode, and  $\vec{B}$  is equilibrium magnetic field.
- Effectively “short-circuit” confinement by allowing heat/particles to radially transit island region by rapidly flowing along magnetic field-lines, rather than slowly diffusing across flux-surfaces.

## Need for Magnetic Island Theory

- Magnetic island formation associated with *nonlinear* phase of tearing mode growth (*i.e.*, when radial island width becomes greater than linear layer width at rational surface).
- In very hot plasmas found in modern-day tokamaks, linear layers so thin that tearing mode already in nonlinear regime when first detected.
- Linear tearing mode theory largely irrelevant. Require nonlinear magnetic island theory to explain experimental observations.

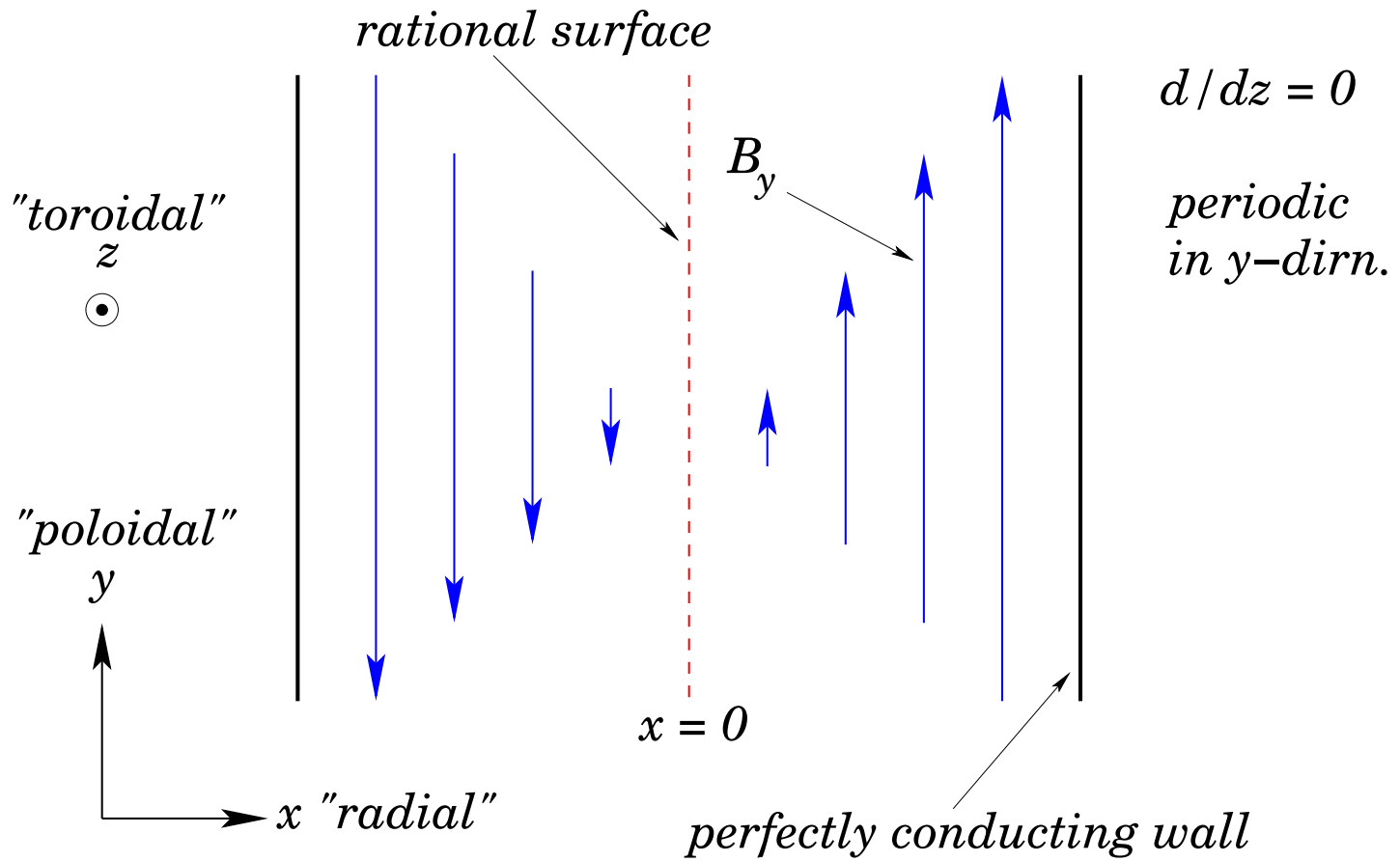
## MHD Theory

- Tearing modes are macroscopic instabilities which affect whole plasma. Natural to investigate them using some form of *fluid-theory*.
- Simplest fluid theory is well-known *magnetohydrodynamical approximation*,<sup>a</sup> which effectively treats plasma as *single-fluid*.
- Shall also use *slab approximation* to simplify analysis.

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<sup>a</sup>*Plasma Confinement*, R.D. Hazeltine, and J.D. Meiss (Dover, 2003).

## *Slab Approximation*



## Slab Model

- Cartesian coordinates:  $(x, y, z)$ . Let  $\partial/\partial z \equiv 0$ .
- Assume presence of dominant uniform “toroidal”  $\vec{B}_z \vec{z}$ .
- All field-strengths normalized to  $B_z$ .
- All lengths normalized to equilibrium magnetic shear-length:

$$L_s = B_z/B'_y(0).$$

- All times normalized to shear-Alfvén time calculated with  $B_z$ .
- Perfect wall boundary conditions at  $x = \pm a$ .
- Wavenumber of tearing instability:  $\vec{k} = (0, k, 0)$ , so  $\vec{k} \cdot \vec{B} = 0$  at  $x = 0$ . Hence, rational surface at  $x = 0$ .

## Model MHD equations

- Let  $\vec{B}_\perp = \nabla\psi \times \vec{z}$  and  $\vec{V} = \nabla\phi \times \vec{z}$ , where  $\vec{V}$  is  $\vec{E} \times \vec{B}$  velocity.
- $\vec{B} \cdot \nabla\psi = \vec{V} \cdot \nabla\phi = 0$ , so  $\psi$  maps magnetic flux-surfaces, and  $\phi$  maps stream-lines of  $\vec{E} \times \vec{B}$  fluid.
- Incompressible MHD equations:<sup>a</sup>

$$\frac{\partial\psi}{\partial t} = [\phi, \psi] + \eta J,$$

$$\frac{\partial\mathcal{U}}{\partial t} = [\phi, \mathcal{U}] + [J, \psi] + \mu \nabla^2 \mathcal{U},$$

where  $J = \nabla^2\psi$ ,  $\mathcal{U} = \nabla^2\phi$ , and  $[A, B] = A_x B_y - A_y B_x$ . Here,  $\eta$  is resistivity, and  $\mu$  is viscosity. In normalized units:  $\eta, \mu \ll 1$ .

- First equation is z-component of Ohm's law. Second equation is z-component of curl of plasma equation of motion.

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<sup>a</sup>*Plasma Confinement*, R.D. Hazeltine, and J.D. Meiss (Dover, 2003).

## Outer Region

- In “outer region”, which comprises most of plasma, non-linear, non-ideal ( $\eta$  and  $\mu$ ), and inertial ( $\partial/\partial t$  and  $\vec{V} \cdot \nabla$ ) effects negligible.
- Vorticity equation reduces to

$$[J, \psi] \simeq 0.$$

- When linearized, obtain  $\psi(x, y) = \psi^{(0)}(x) + \psi^{(1)}(x) \cos(ky)$ , where  $B_y^{(0)} = -d\psi^{(0)}/dx$ , and

$$\left( \frac{d^2}{dx^2} - k^2 \right) \psi^{(1)} - \left( \frac{d^2 B_y^{(0)}/dx^2}{B_y^{(0)}} \right) \psi^{(1)} = 0.$$

- Equation is *singular* at rational surface,  $x = 0$ , where  $B_y^{(0)} = 0$ .

## Tearing Stability Index

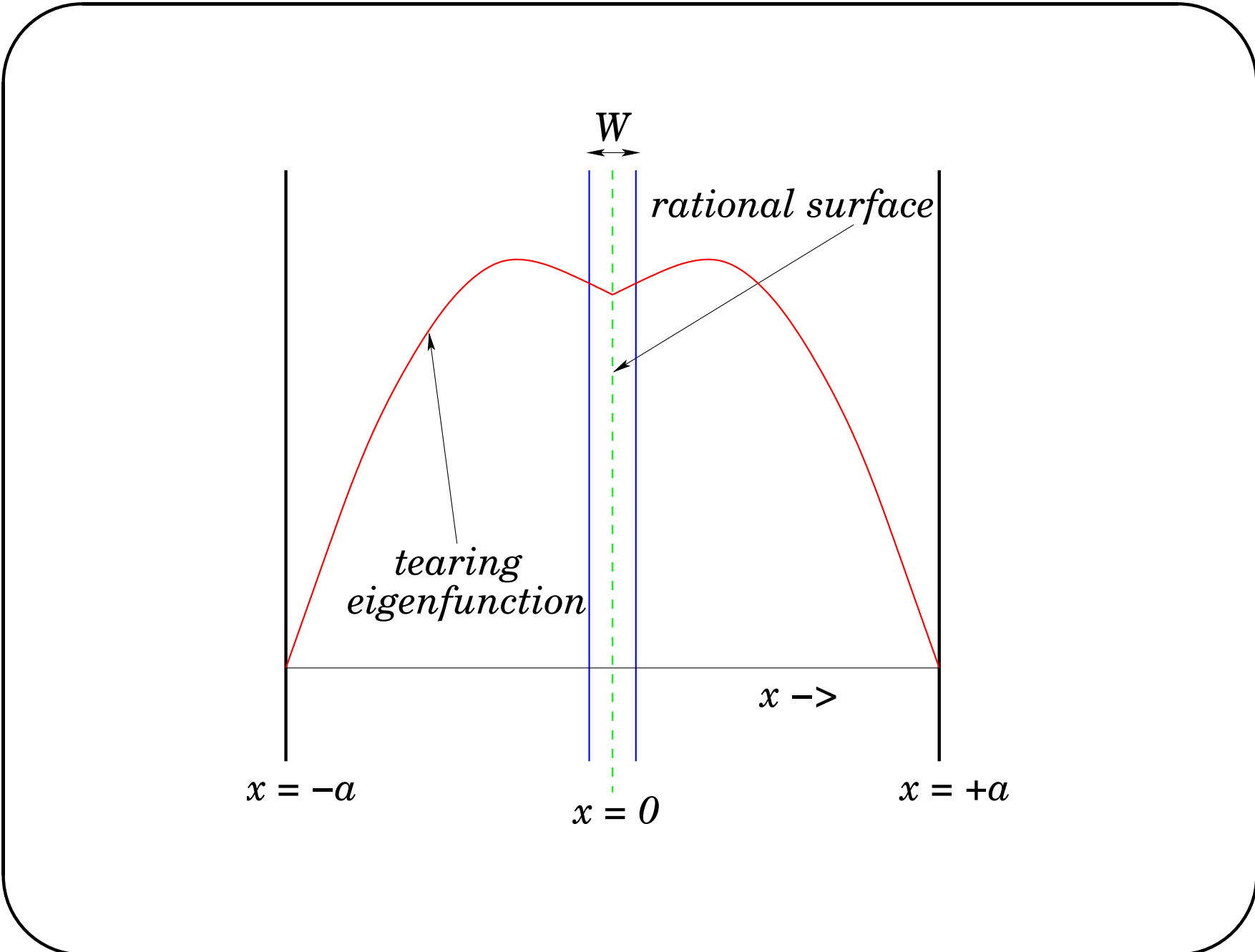
- Find tearing eigenfunction,  $\psi^{(1)}(x)$ , which is continuous, has tearing parity [ $\psi^{(1)}(-x) = \psi^{(1)}(x)$ ], and satisfies boundary condition  $\psi^{(1)}(a) = 0$  at conducting wall.
- In general, eigenfunction has *gradient discontinuity* across rational surface (at  $x = 0$ ). Allowed because tearing mode equation singular at rational surface.
- Tearing stability index:

$$\Delta' = \left[ \frac{d \ln \psi^{(1)}}{dx} \right]_{0-}^{0+}.$$

- According to conventional MHD theory,<sup>a</sup> tearing mode is unstable if  $\Delta' > 0$ .

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<sup>a</sup>H.P. Furth, J. Killeen, and M.N. Rosenbluth, Phys. Fluids **6**, 459 (1963).



## Inner Region

- “Inner region” centered on rational surface,  $x = 0$ . Of extent,  $W \ll 1$ , where  $W$  is magnetic island width (in  $x$ ).
- In inner region, non-ideal effects, non-linear effects, and plasma inertia can all be important.
- Inner solution must be asymptotically matched to outer solution already obtained.

## Constant- $\psi$ Approximation

- $\psi^{(1)}(x)$  generally does not vary significantly in  $x$  over inner region:

$$|\psi^{(1)}(W) - \psi^{(1)}(0)| \ll |\psi^{(1)}(0)|.$$

- *Constant- $\psi$  approximation*: treat  $\psi^{(1)}(x)$  as constant in  $x$  over inner region.
- Approximation valid provided

$$|\Delta'|W \ll 1,$$

which is easily satisfied for conventional tearing modes.

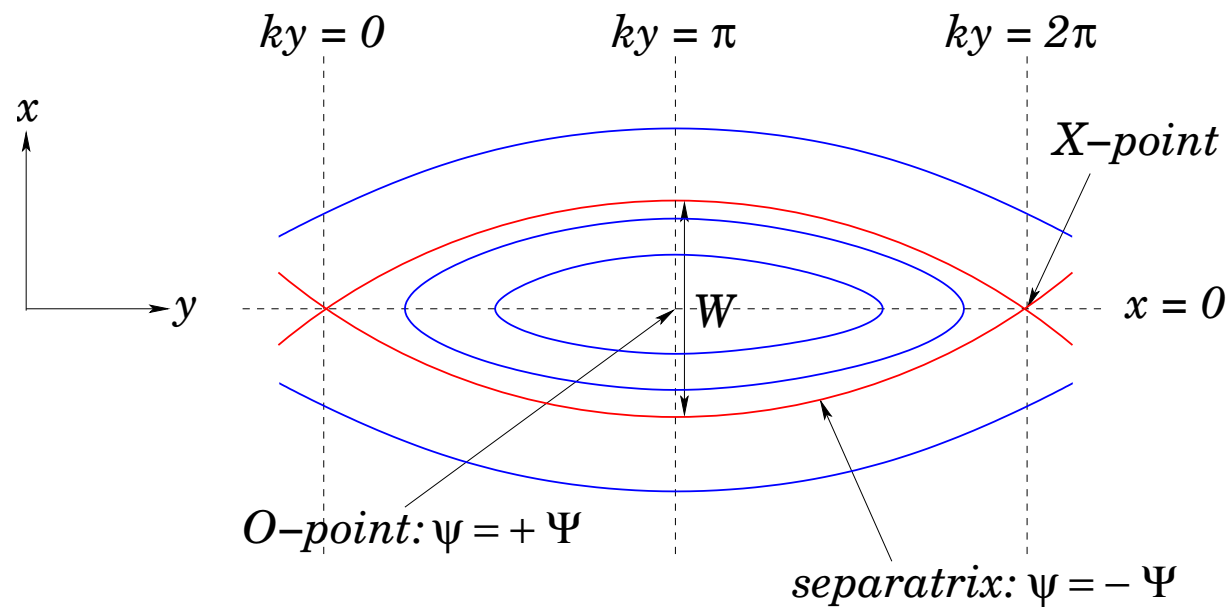
## Constant- $\psi$ Magnetic Island

- In vicinity of rational surface,  $\psi^{(0)} \rightarrow -x^2/2$ , so

$$\psi(x, y, t) \simeq -x^2/2 + \Psi(t) \cos \theta,$$

where  $\Psi = \psi^{(1)}(0)$  is “reconnected flux”, and  $\theta = ky$ .

- Full island width,  $W = 4 \sqrt{\Psi}$ .



## Flux-Surface Average Operator

- Flux-surface average operator is annihilator of Poisson bracket  
 $[A, \psi] \equiv \vec{B} \cdot \nabla A \equiv k \times (\partial A / \partial \theta)_\psi$  for any  $A$ : *i.e.*,

$$\langle [A, \psi] \rangle \equiv 0.$$

- Outside separatrix:

$$\langle f(\psi, \theta) \rangle = \oint \frac{f(\psi, \theta)}{|\mathbf{x}|} \frac{d\theta}{2\pi}.$$

- Inside separatrix:

$$\langle f(s, \psi, \theta) \rangle = \int_{-\theta_0}^{\theta_0} \frac{f(s, \psi, \theta) + f(-s, \psi, \theta)}{2|\mathbf{x}|} \frac{d\theta}{2\pi},$$

where  $s = \text{sgn}(\mathbf{x})$ , and  $\mathbf{x}(s, \psi, \theta_0) = 0$ .

## MHD Flow -I

- Move to island frame. Look for steady-state solution:  $\partial/\partial t = 0$ .<sup>a</sup>
- Ohm's law:

$$0 \simeq [\phi, \psi] + \eta J.$$

- Since  $\eta \ll 1$ , first term potentially much larger than second.
- To lowest order:

$$[\phi, \psi] \simeq 0.$$

- Follows that

$$\phi = \phi(\psi) :$$

*i.e.*, MHD flow constrained to be around flux-surfaces.

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<sup>a</sup>F.L. Waelbroeck, and R. Fitzpatrick, Phys. Rev. Lett. **78**, 1703 (1997).

## MHD Flow - II

- Let

$$M(\psi) = \frac{d\phi}{d\psi}.$$

- Easily shown that

$$V_y = x M.$$

- By symmetry,  $M(\psi)$  is *odd* function of  $x$ . Hence,

$$M = 0$$

inside separatrix: *i.e.*, no flow inside separatrix in island frame.  
Plasma *trapped* within magnetic separatrix.

## MHD Flow - III

- Vorticity equation:

$$0 \simeq [-M \mathbf{U} + \mathbf{J}, \psi] + \mu \nabla^4 \phi.$$

- Flux-surface average, recalling that  $\langle [A, \psi] \rangle = 0$ :

$$\langle \nabla^4 \phi \rangle \equiv -\frac{d^2}{d\psi^2} \left( \langle x^4 \rangle \frac{dM}{d\psi} \right) \simeq 0.$$

- Solution outside separatrix:

$$M(\psi) = \text{sgn}(x) M_0 \int_{-\Psi}^{\psi} d\psi / \langle x^4 \rangle \bigg/ \int_{-\Psi}^{-\infty} d\psi / \langle x^4 \rangle.$$

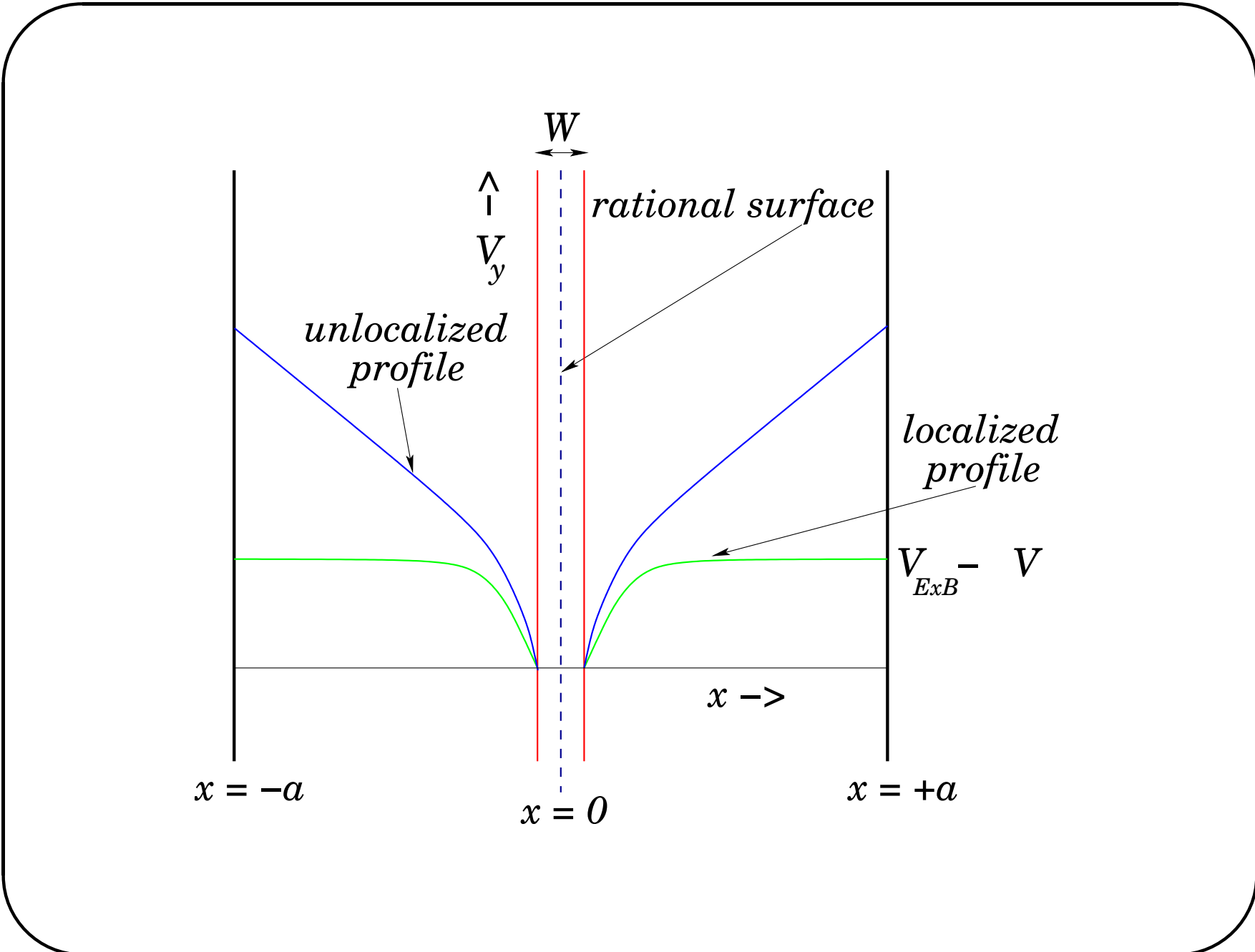
## MHD Flow - IV

- Note that

$$V_y = x M \rightarrow |x| M_0$$

as  $|x|/W \rightarrow \infty$ .

- V-shaped velocity profile which extends over whole plasma.
- Expect *isolated* magnetic island to have *localized* velocity profile. Suggests that  $M_0 = 0$  for isolated island.
- Hence, zero MHD flow in island frame: *i.e.*, island propagates at local  $\vec{E} \times \vec{B}$  velocity.



## Rutherford Equation - I

- Asymptotic matching between inner and outer regions yields:

$$\Delta' \Psi = -4 \int_{+\Psi}^{-\infty} \langle J \cos \theta \rangle d\psi.$$

- In island frame, in absence of MHD flow, vorticity equation reduces to

$$[J, \psi] \simeq 0.$$

- Hence,

$$J = J(\psi).$$

## Rutherford Equation - II

- Ohm's law:

$$\frac{d\Psi}{dt} \cos \theta \simeq [\phi, \psi] + \eta J(\psi).$$

- Have shown there is no MHD-flow [*i.e.*,  $\phi \sim O(1)$ ], but can still be *resistive flow* [*i.e.*,  $\phi \sim O(\eta)$ ].
- Eliminate resistive flow by flux-surface averaging:

$$\frac{d\Psi}{dt} \langle \cos \theta \rangle \simeq \eta J(\psi) \langle 1 \rangle.$$

- Hence,

$$\Delta' \Psi \simeq -\frac{4}{\eta} \frac{d\Psi}{dt} \int_{+\Psi}^{-\infty} \frac{\langle \cos \theta \rangle^2}{\langle 1 \rangle} d\psi.$$

## Rutherford Equation - III

- Use  $W = 4 \sqrt{\Psi}$ , and evaluate integral. Obtain *Rutherford island width evolution equation*:<sup>a</sup>

$$\frac{0.823}{\eta} \frac{dW}{dt} \simeq \Delta'.$$

- According to Rutherford equation, island grows *algebraically* on *resistive time-scale*.
- Rutherford equation does not predict island saturation.

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<sup>a</sup>P.H. Rutherford, Phys. Fluids **16**, 1903 (1973).

## Rutherford Equation - IV

- Higher order asymptotic matching between inner and outer regions yields: <sup>a</sup>

$$\frac{0.823}{\eta} \frac{dW}{dt} \simeq \Delta' - 0.41 \left( -\frac{d^4 B_y^{(0)} / dx^4}{d^2 B_y^{(0)} / dx^2} \right)_{x=0} W.$$

- Hence, saturated ( $d/dt = 0$ ) island width is

$$W_0 = \frac{\Delta'}{0.41} \left( -\frac{d^2 B_y^{(0)} / dx^2}{d^4 B_y^{(0)} / dx^4} \right)_{x=0}.$$

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<sup>a</sup>F. Militello, and F. Porcelli, Phys. Plasmas **11**, L13 (2004). D.F. Escande, and M. Ottaviani, Physics Lett. A **323**, 278 (2004).

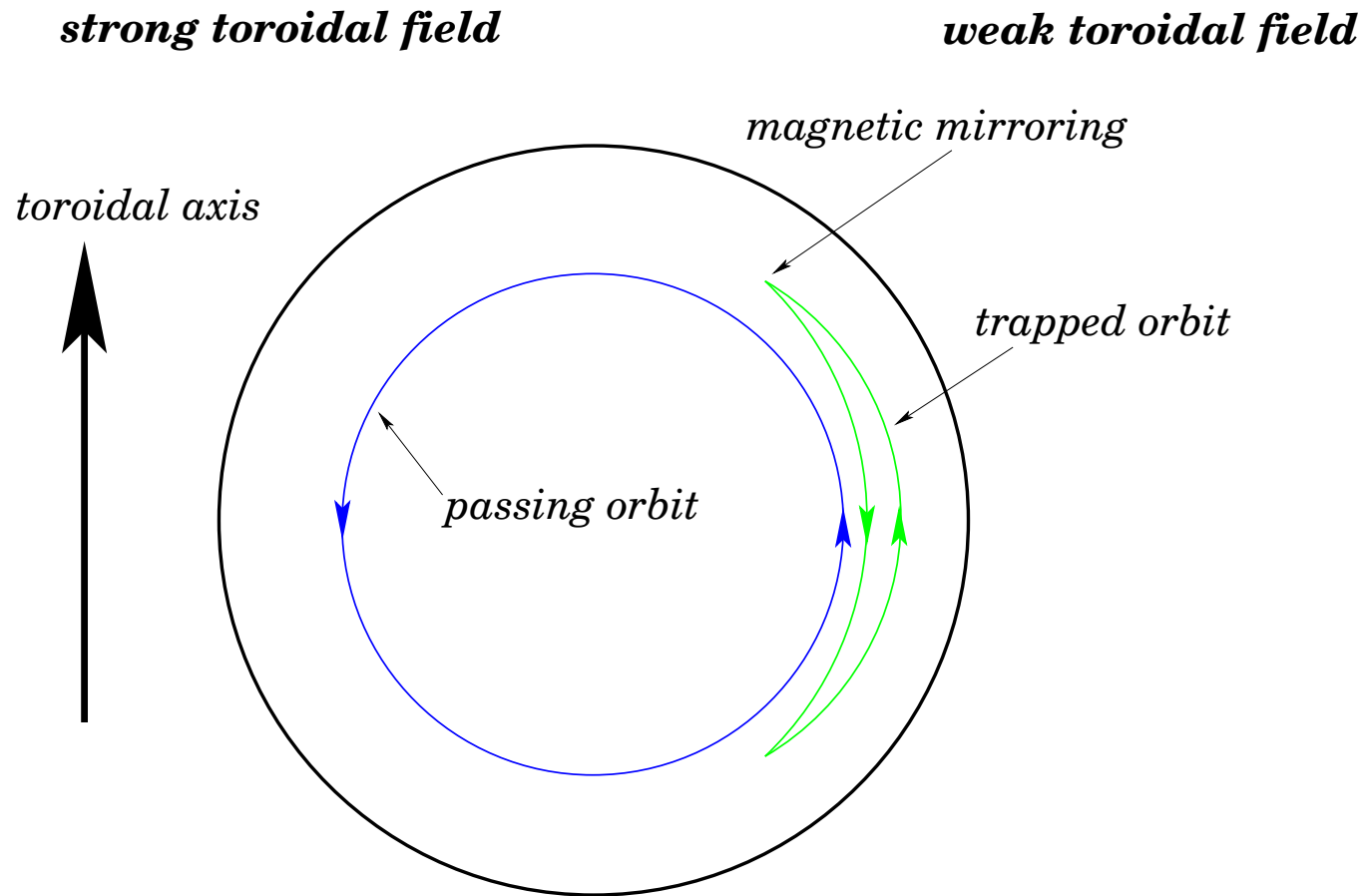
## Neoclassical Effects

- So-called *neoclassical effects*<sup>a</sup> in tokamaks arise from combination of *toroidal geometry*, and *extremely long mean-free-path* of electrons and ions streaming along field-lines, due to very low collisionality of hot fusion plasmas.

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<sup>a</sup>*The Theory of Toroidally Confined Plasmas*, 2nd Rev. Edition, R.B. White (World Scientific, 2006).

# Trapped and Passing Particles



## Bootstrap Current - I

- In tokamak plasma, friction between trapped and passing electrons leads to appearance of non-inductive *bootstrap current* in Ohm's law: <sup>a</sup>

$$\frac{d\Psi}{dt} \cos \theta \simeq [\phi, \psi] + \eta [J(\psi) - J_{\text{boot}}],$$

where

$$J_{\text{boot}} = -1.46 \sqrt{\epsilon} B_{\theta}^{-1} \frac{\partial P}{\partial r}.$$

Here,  $\epsilon$  is inverse aspect-ratio,  $1.46 \sqrt{\epsilon}$  is measure of fraction of trapped-particles, and  $P$  is plasma pressure.

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<sup>a</sup>M.N. Rosenbluth, R.D. Hazeltine, and F.L. Hinton, Phys. Fluids **15**, 116 (1972).

## Bootstrap Current - II

- Pressure profile *flattened* inside island separatrix due to fast parallel transport.
- Bootstrap current consequently disappears inside separatrix.
- Absence of bootstrap current inside separatrix, and continued presence outside, leads to *destabilizing* term in Rutherford island equation:<sup>a</sup>

$$\frac{0.823}{\eta} \frac{dW}{dt} \simeq \Delta' - 2.31 \sqrt{\epsilon} \frac{(r P' / B_{\theta}^2)}{(W/4)}$$

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<sup>a</sup>R. Fitzpatrick, Phys. Plasmas **2**, 825 (1995).

## Neoclassical Tearing Modes - I

- A *neoclassical tearing mode* (NTM) is an *intrinsically stable* ( $\Delta' < 0$ ) tearing mode destabilized by bootstrap term.
- Bootstrap term in Rutherford equation relatively large, especially at small island widths. Would expect plasma to be filled with NTMs, and confinement to be wrecked.
- This is not observed to be case. Experimental evidence for *threshold island width* above which NTMs grow, but below which they decay.<sup>a</sup>
- Suggests presence of *stabilizing term* in Rutherford equation which opposes destabilizing bootstrap term.

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<sup>a</sup>O. Sauter, *et al.*, Phys. Plasmas **4**, 1654 (1997).

## Neoclassical Tearing Modes - II

- Most likely candidate for stabilizing term in Rutherford equation, which provides NTM threshold mechanism, is well-known term due to **ion polarization current**.<sup>a</sup>
- In order to investigate this term, must graduate to **two-fluid** drift-MHD magnetic island theory.

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<sup>a</sup>A.I. Smolyakov, Sov. J. Plasma Phys. **15**, 667 (1989).

## Drift-MHD Theory

- In drift-MHD approximation, analysis retains *charged particle drift velocities*, in addition to  $\vec{E} \times \vec{B}$  velocity.
- Essentially *two-fluid* theory of plasma.
- Characteristic length-scale,  $\rho$ , is *ion Larmor radius calculated with electron temperature*.
- Characteristic velocity is *diamagnetic velocity*,  $V_*$ , where

$$n e \vec{V}_* \times \vec{B} = \nabla P.$$

- Normalize all lengths to  $\rho$ , and all velocities to  $V_*$ .

## Basic Assumptions

- Retain slab model, for sake of simplicity.
- Assume parallel electron heat transport sufficiently strong that  $T_e = T_e(\psi)$ .
- Assume  $T_i/T_e = \tau = \text{constant}$ , for sake of simplicity.

## Basic Definitions

- Variables:
  - $\psi$  - magnetic flux-function.
  - $J$  - parallel current.
  - $\phi$  - guiding-center (*i.e.*, MHD) stream-function.
  - $\mathcal{U}$  - parallel ion vorticity.
  - $n$  - electron number density (minus uniform background).
  - $V_z$  - parallel ion velocity.
- Parameters:
  - $\alpha = (L_n/L_s)^2$ , where  $L_n$  is equilibrium density gradient scale-length.
  - $\eta$  - resistivity.  $D$  - (perpendicular) particle diffusivity.  $\mu_{i/e}$  - (perpendicular) ion/electron viscosity.

## Drift-MHD Equations - I

- Steady-state drift-MHD equations: <sup>a</sup>

$$\psi = -x^2/2 + \Psi \cos \theta, \quad \mathbf{U} = \nabla^2 \phi,$$

$$0 = [\phi - n, \psi] + \eta J,$$

$$0 = [\phi, \mathbf{U}] - \frac{\tau}{2} \{ \nabla^2 [\phi, n] + [\mathbf{U}, n] + [\nabla^2 n, \phi] \} \\ + [J, \psi] + \mu_i \nabla^4 (\phi + \tau n) + \mu_e \nabla^4 (\phi - n),$$

$$0 = [\phi, n] + [V_z + J, \psi] + D \nabla^2 n,$$

$$0 = [\phi, V_z] + \alpha [n, \psi] + \mu_i \nabla^2 V_z.$$

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<sup>a</sup>R.D. Hazeltine, M. Kotschenreuther, and P.J. Morrison, Phys. Fluids **28**, 2466 (1985).

## Drift-MHD Equations - II

- Symmetry:  $\psi, J, V_z$  even in  $x$ .  $\phi, n, U$  odd in  $x$ .
- Boundary conditions as  $|x|/W \rightarrow \infty$ :
  - $n \rightarrow -(1 + \tau)^{-1} x$ .
  - $\phi \rightarrow -V x$ .
  - $J, U, V_z \rightarrow 0$ .
- Here,  $V$  is island phase-velocity in  $\vec{E} \times \vec{B}$  frame.
- $V = 1$  corresponds to island propagating with electron fluid.  
 $V = -\tau$  corresponds to island propagating with ion fluid.
- Expect

$$1 \gg \alpha \gg \eta, D, \mu_i, \mu_e.$$

## Electron Fluid

- Ohm's law:

$$0 = [\phi - n, \psi] + \eta J.$$

- Since  $\eta \ll 1$ , first term potentially much larger than second.
- To lowest order:

$$[\phi - n, \psi] \simeq 0.$$

- Follows that

$$n = \phi + H(\psi) :$$

*i.e.*, electron stream-function  $\phi_e = \phi - n$  is *flux-surface function*.  
Electron fluid flow constrained to be around flux-surfaces.

## Sound Waves

- Parallel flow equation:

$$0 = [\phi, V_z] + \alpha [n, \psi] + \mu_i \nabla^2 V_z.$$

- Highlighted term dominant provided

$$W \gg \alpha^{-1/2} = L_s/L_n.$$

- If this is case then to lowest order

$$n = n(\psi),$$

which implies  $n = 0$  inside separatrix.

- So, if island sufficiently wide, *sound-waves* able to *flatten density profile* inside island separatrix.

## Subsonic vs. Supersonic Islands

- Wide islands satisfying

$$W \gg L_s/L_n$$

termed *subsonic* islands. Expect such islands to exhibit flattened density profile within separatrix. Subsonic islands strongly coupled to both electron and ion fluids.

- Narrow islands satisfying

$$W \ll L_s/L_n$$

termed *supersonic* islands. No flattening of density profile within separatrix. Supersonic islands strongly coupled to electron fluid, but only weakly coupled to ion fluid.

## Subsonic Islands<sup>a</sup>

- To lowest order:

$$\phi = \phi(\psi), \quad n = n(\psi).$$

- Follows that both electron stream-function,  $\phi_e = \phi - n$ , and ion stream-function,  $\phi_i = \phi + \tau n$ , are flux-surface functions. Both electron and ion fluid flow constrained to follow flux-surfaces.
- Let

$$M(\psi) = d\phi/d\psi, \quad L(\psi) = dn/d\psi.$$

- Follows that

$$V_{E \times B y} = \chi M, \quad V_{e y} = \chi (M - L), \quad V_{i y} = \chi (M + \tau L).$$

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<sup>a</sup>R. Fitzpatrick, F.L. Waelbroeck, Phys. Plasmas **12**, 022307 (2005).

## Density Flattening

- By symmetry, both  $M(\psi)$  and  $L(\psi)$  are *odd* functions of  $x$ .  
Hence,

$$M(\psi) = L(\psi) = 0$$

inside separatrix: *i.e.*, no electron/ion flow within separatrix in island frame.

- Electron/ion fluids constrained to propagate with island inside separatrix.
- Density profile *flattened* within separatrix.

## Analysis - I

- Density equation reduces to

$$0 \simeq [V_z + J, \psi] + D \nabla^2 n.$$

- Vorticity equation reduces to

$$0 \simeq [-M U - (\tau/2)(L U + M \nabla^2 n) + J, \psi] \\ + \mu_i \nabla^4 (\phi + \tau n) + \mu_e \nabla^4 (\phi - n).$$

- Flux-surface average both equations, recalling that  $\langle [A, \psi] \rangle = 0$ .

## Analysis - II

- Obtain

$$\langle \nabla^2 \mathbf{n} \rangle \simeq 0,$$

and

$$(\mu_i + \mu_e) \langle \nabla^4 \phi \rangle + (\mu_i \tau - \mu_e) \langle \nabla^4 \mathbf{n} \rangle \simeq 0.$$

- Solution outside separatrix:

$$M(\psi) = -\frac{(\mu_i \tau - \mu_e)}{(\mu_i + \mu_e)} L(\psi) + F(\psi),$$

where

$$L(\psi) = -\text{sgn}(x) L_0 / \langle x^2 \rangle,$$

and  $F(\psi)$  is previously obtained MHD profile:

$$F(\psi) = \text{sgn}(x) F_0 \int_{-\Psi}^{\psi} d\psi / \langle x^4 \rangle \bigg/ \int_{-\Psi}^{-\infty} d\psi / \langle x^4 \rangle.$$

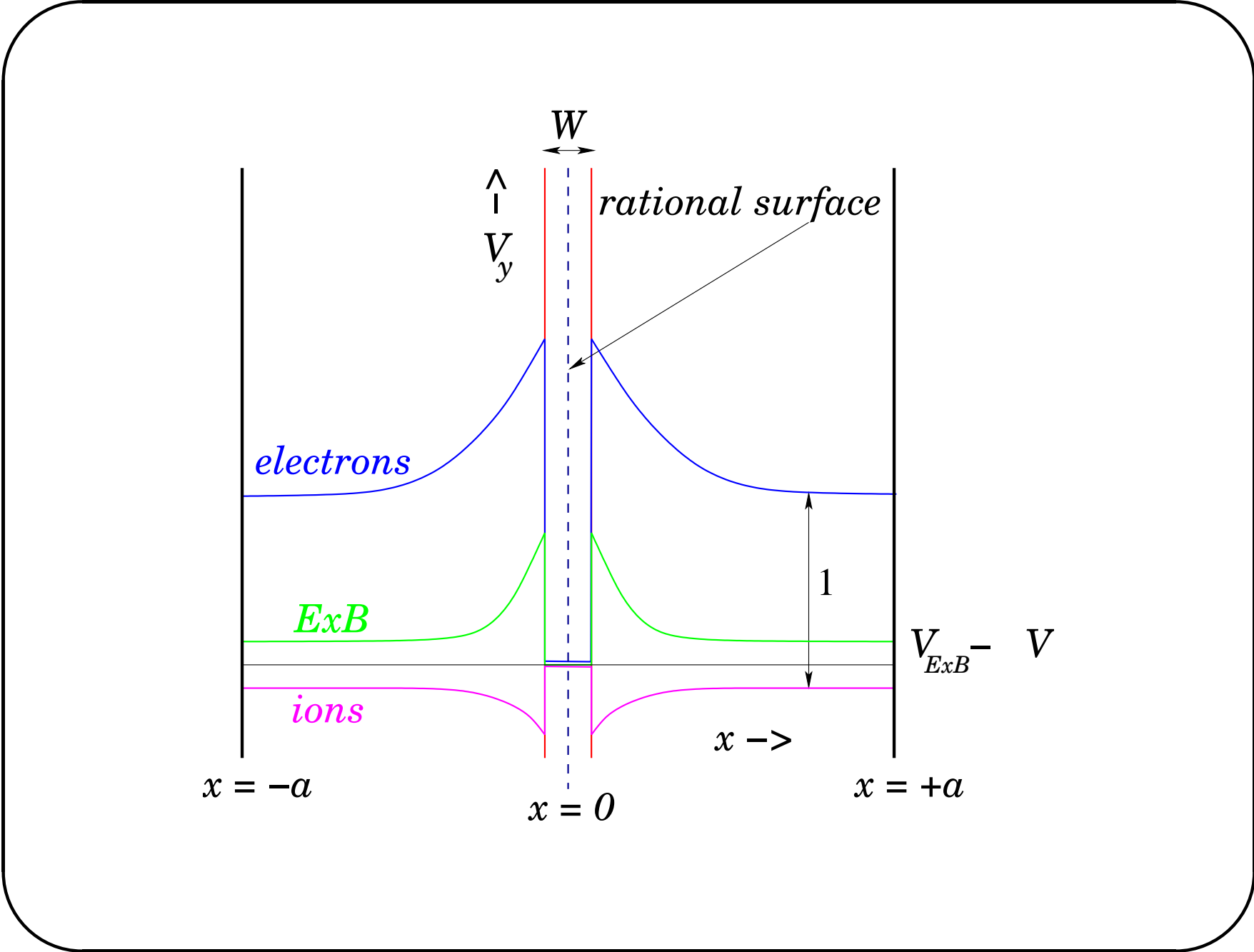
## Velocity Profiles

- As  $|x|/W \rightarrow \infty$  then  $x L \rightarrow L_0$  and  $x F \rightarrow |x| F_0$ .
- $L(\psi)$  corresponds to *localized* velocity profile.  $F(\psi)$  corresponds to *non-localized* profile. Require localized profile, so  $F_0 = 0$ .
- Velocity profiles outside separatrix (using b.c. on  $n$ ):

$$V_{y i} \simeq + \frac{\mu_e}{\mu_i + \mu_e} \frac{|x|}{\langle x^2 \rangle},$$

$$V_{y E \times B} \simeq - \frac{(\mu_i \tau - \mu_e)}{(1 + \tau)(\mu_i + \mu_e)} \frac{|x|}{\langle x^2 \rangle},$$

$$V_{y e} = - \frac{\mu_i}{\mu_i + \mu_e} \frac{|x|}{\langle x^2 \rangle}.$$



## Island Propagation

- As  $|x|/W \rightarrow \infty$  expect  $V_{y \text{ E} \times \text{B}} \rightarrow V_{\text{EB}} - V$ , where  $V_{\text{EB}}$  is unperturbed (*i.e.*, no island)  $\vec{E} \times \vec{B}$  velocity at rational surface (in lab. frame), and  $V$  is island phase-velocity (in lab. frame).

- Hence

$$V = V_{\text{EB}} + \frac{(\mu_i \tau - \mu_e)}{(1 + \tau)(\mu_i + \mu_e)}.$$

- But unperturbed ion/electron fluid velocities (in lab. frame):

$$V_i = V_{\text{EB}} + \tau/(1 + \tau), \quad V_e = V_{\text{EB}} - 1/(1 + \tau).$$

- Hence

$$V = \frac{\mu_i}{\mu_i + \mu_e} V_i + \frac{\mu_e}{\mu_i + \mu_e} V_e.$$

So, island phase-velocity is *viscosity weighted average* of unperturbed ion/electron fluid velocities.

## Polarization Term - I

- Vorticity equation yields

$$J_c \simeq \frac{1}{2} \left( x^2 - \frac{\langle x^2 \rangle}{\langle 1 \rangle} \right) \frac{d[M (M + \tau L)]}{d\psi} + I(\psi)$$

outside separatrix, where  $J_c$  is part of  $J$  with  $\cos \theta$  symmetry.

- As before, flux-surface average of Ohm's law yields:

$$\langle J_c \rangle = I(\psi) \langle 1 \rangle = \eta^{-1} \frac{d\Psi}{dt} \langle \cos \theta \rangle.$$

- Hence

$$J_c \simeq \frac{1}{2} \left( x^2 - \frac{\langle x^2 \rangle}{\langle 1 \rangle} \right) \frac{d[M (M + \tau L)]}{d\psi} + \eta^{-1} \frac{d\Psi}{dt} \frac{\langle \cos \theta \rangle}{\langle 1 \rangle}.$$

## Polarization Term - II

- Asymptotic matching between inner and outer regions yields:

$$\Delta' \Psi = -4 \int_{+\Psi}^{-\infty} \langle J_c \cos \theta \rangle d\psi.$$

- Evaluating flux-surface integrals, making use of previous solutions for M and L, obtain modified Rutherford equation:

$$\frac{0.823}{\eta} \frac{dW}{dt} \simeq \Delta' + 1.38 \beta \frac{(V - V_{EB})(V - V_i)}{(W/4)^3}.$$

- New term is due to *polarization current* associated with ion fluid flow around curved island flux-surfaces (in island frame). Obviously, new term is zero if island propagates with ion fluid: *i.e.*,  $V = V_i$ .

## MHD Theory: Summary

- Tearing mode unstable if  $\Delta' > 0$ .
- Island propagates at local  $\vec{E} \times \vec{B}$  velocity at rational surface.
- Island grows algebraically on resistive time-scale.
- Saturated island width:

$$W_0 = \frac{\Delta'}{0.41} \left( -\frac{d^2 B_y^{(0)} / dx^2}{d^4 B_y^{(0)} / dx^4} \right)_{x=0} .$$

## Drift-MHD Theory: Summary

- Results limited to large islands: *i.e.*, large enough for sound waves to flatten density profile.
- Island propagates at (perpendicular) viscosity weighted average of unperturbed (no island) ion and electron fluid velocities.
- Bootstrap term in Rutherford equation is **destabiizing**.
- Polarization term in Rutherford equation is **stabilizing** provided ion (perpendicular) viscosity greatly exceeds electron (perpendicular) viscosity (which is what we expect), and destabilizing otherwise.