Calculation of Tearing Mode Stability in Tokamak Plasmas Via Asymptotic Matching

Richard Fitzpatrick

Institute for Fusion Studies

University of Texas at Austin

Austin TX, USA

Global Resistive Stability of Tokamak Plasmas

- Determination of global resistive stability of tokamak plasma can be reduced to *asymptotic matching* problem.[a]
- System divided into *inner region* that is highly localized in vicinity of rational flux surfaces, and *outer region* that comprises remainder of plasma (and any surrounding vacuum).
- Linearised, marginally stable, ideal-MHD equations solved in outer region. Solutions asymptotically matched to linear/nonlinear resistive-MHD layer solutions at rational surfaces.
- Matching process leads to matrix dispersion relation.[b]

---

[a] H.P. Furth, J. Killeen, and M.N. Rosenbluth, Phys. Fluids **6**, 459 (1963).
[b] J.W. Connor, et al., Phys. Fluids **31**, 577 (1988).

Tomuhawc Code

- Recently developed TOMUHAWC code calculates those elements of matrix dispersion relation that depend on ideal-MHD solution in outer region.
- Ideal-MHD equations are converted into a set of coupled first-order differential equations in flux-surface label. These equations are solved via adaptive-step integration.
- TOMUHAWC capable of using metric data generated by general equilibrium code.

Coordinates

- All lengths normalized to plasma minor radius, a, all magnetic field-strengths to vacuum field-strength at magnetic axis, B_0, and all plasma pressures to B_0^2/μ_0.

- TOMUHAWC employs flux coordinate system r, θ, ϕ. r is flux-surface label that is zero at magnetic axis and unity at plasma boundary. θ is straight poloidal angle. ϕ is geometric toroidal angle. Jacobian:

$$(\nabla r \times \nabla\theta \cdot \nabla\phi)^{-1} = r\,R_0 \left(\frac{R}{R_0}\right)^2,$$

  where R is major radius, and R_0 is major radius of magnetic axis.

Plasma Equilibrium

- Equilibrium magnetic field:

$$\mathbf{B}(r,\theta) = \frac{r\,g}{q}\,\nabla(\phi - q\,\theta) \times \nabla r,$$

where $q(r)$ is safety-factor profile, and $g(r)$ determines toroidal magnetic field-strength.

- Equilibrium determined from Grad-Shafranov equation,

$$\frac{g}{q}\,\frac{\partial}{\partial r}\left(r^2\,\frac{g}{q}\,|\nabla r|^{\,2}\right)+\frac{g}{q}\,\frac{\partial}{\partial \theta}\left(r^2\,\frac{g}{q}\,\nabla r\cdot\nabla\theta\right)+R_0^{\,2}\,g\,\frac{dg}{dr}+R^{\,2}\,\frac{dp}{dr} = 0,$$

where $p(r)$ is plasma pressure.

Profile Functions

- TOMUHAWC requires following six profile functions:

$$\begin{aligned}
p_1(r) &= q, \\
p_2(r) &= \frac{dq}{dr}, \\
p_3(r) &= \left(\frac{r}{R_0}\right)^2, \\
p_4(r) &= \left(\frac{q}{g}\right)\frac{R_0^2}{r}\frac{dg}{dr}, \\
p_5(r) &= \left(\frac{q}{g}\right)^2\frac{R_0^2}{r}\frac{dp}{dr}, \\
p_6(r) &= \left(\frac{q}{g}\right) r \frac{d}{dr}\left(\frac{g}{q}\right).
\end{aligned}$$

Metric Functions

- TOMUHAWC requires following metric functions:

$$
\begin{aligned}
M^{(1)}_{jj'}(r) &= \oint \left(\frac{R}{R_0}\right)^2 \exp\left[-i\left(m_j - m_{j'}\right)\theta\right] \frac{d\theta}{2\pi}, \\
M^{(2)}_{jj'}(r) &= \oint |\nabla r|^{-2} \left(\frac{R}{R_0}\right)^{-2} \exp\left[-i\left(m_j - m_{j'}\right)\theta\right] \frac{d\theta}{2\pi}, \\
M^{(3)}_{jj'}(r) &= \oint |\nabla r|^{-2} \exp\left[-i\left(m - m'\right)\theta\right] \frac{d\theta}{2\pi}, \\
M^{(4)}_{jj'}(r) &= \oint |\nabla r|^{-2} \left(\frac{R}{R_0}\right)^{2} \exp\left[-i\left(m_j - m_{j'}\right)\theta\right] \frac{d\theta}{2\pi}, \\
M^{(5)}_{jj'}(r) &= \oint |\nabla r|^{-2} \left(\frac{R}{R_0}\right)^{4} \exp\left[-i\left(m_j - m_{j'}\right)\theta\right] \frac{d\theta}{2\pi}, \\
M^{(6)}_{jj'}(r) &= \oint \frac{i\, r\, \nabla r \cdot \nabla\theta}{|\nabla r|^2} \exp\left[-i\left(m_j - m_{j'}\right)\theta\right] \frac{d\theta}{2\pi}, \\
M^{(7)}_{jj'}(r) &= \oint \frac{i\, r\, \nabla r \cdot \nabla\theta}{|\nabla r|^2} \left(\frac{R}{R_0}\right)^{2} \exp\left[-i\left(m_j - m_{j'}\right)\theta\right] \frac{d\theta}{2\pi}.
\end{aligned}
$$

Rational Surfaces

- Rational surfaces satisfy $q(r_k) = m_k/n$, where n is toroidal mode number, and m_k is one of coupled poloidal mode numbers.

- Write

$$\delta\mathbf{B}\cdot\nabla r = i\left(\frac{R_0}{R}\right)^2 \sum_{j=1,J} \frac{\psi_j(r)}{r}\,\exp[i\,(m_j\,\theta - n\,\phi)].$$

- Ideal-MHD solution in immediate vicinity of kth rational surface such that

$$\psi_k(r_k + x) = A^{\pm}_{L\,k}\,|x|^{\nu_{L\,k}} + A^{\pm}_{S\,k}\,\mathrm{sgn}(x)\,|x|^{\nu_{S\,k}} + A_{C\,k}\,x + \cdots,$$

where $\nu_{L\,k} = 1/2 - \sqrt{D_{I\,k}}$, $\nu_{S\,k} = 1/2 + \sqrt{D_{I\,k}}$, and $D_{I\,k}$ is standard ideal interchange stability parameter at surface.[a]

[a] A.H. Glasser, J.M. Greene, and J.L. Johnson, Phys. Fluids **18**, 875 (1975).

Matching Parameters

$$\Delta_k^e = 2\, r_k^{2\sqrt{D_{I\,k}}} \left(\frac{A_{S\,k}^+ - A_{S\,k}^-}{A_{L\,k}^+ + A_{L\,k}^-} \right),$$

$$\Delta_k^o = 2\, r_k^{2\sqrt{D_{I\,k}}} \left(\frac{A_{S\,k}^+ + A_{S\,k}^-}{A_{L\,k}^+ - A_{L\,k}^-} \right),$$

$$\Psi_k^e = \frac{1}{2}\, r_k^{2\sqrt{D_{I\,k}}} \left(\frac{2\sqrt{D_{I\,k}}}{m_k^2\, \langle |\nabla r|^2\rangle + n^2\, (r/R_0)^2} \right)_{r_k}^{1/2} (A_{L\,k}^+ + A_{L\,k}^-),$$

$$\Psi_k^o = \frac{1}{2}\, r_k^{2\sqrt{D_{I\,k}}} \left(\frac{2\sqrt{D_{I\,k}}}{m_k^2\, \langle |\nabla r|^2\rangle + n^2\, (r/R_0)^2} \right)_{r_k}^{1/2} (A_{L\,k}^+ - A_{L\,k}^-).$$

- Δ_k^e and Δ_k^o determined by tearing and twisting parity layer solutions at kth rational surface. Ψ_k^e and Ψ_k^o parameterize magnetic reconnection at kth surface due to tearing and twisting parity modes. (So, Ψ_k^e and Ψ_k^o are zero in ideal plasma.)

Dispersion Relation

- Asymptotic matching leads to matrix dispersion relation:

$$\Delta_k^e \Psi_k^e = \sum_{k'=1,K} \left(E^e_{kk'} \Psi^e_{k'} + \Gamma_{kk'} \Psi^o_{k'}\right),$$

$$\Delta_k^o \Psi_k^o = \sum_{k'=1,K} \left(E^o_{kk'} \Psi^o_{k'} + \Gamma'_{kk'} \Psi^e_{k'}\right).$$

  Here, K is number of coupled rational surfaces in plasma.

- TOMUHAWC calculates elements of $\mathbf{E}^e$, $\mathbf{E}^o$, $\mathbf{\Gamma}$, and $\mathbf{\Gamma}'$ matrices. These elements only depend on numerical solution of linearized, marginally stable, ideal-MHD equations in outer region.

Electromagnetic Torques

- Zero flux-surface averaged toroidal electromagnetic torque exerted on plasma in outer region.
- Net torque exerted on segment of inner region at kth rational surface is

$$\delta T_k = 2\,n\,\pi^2\,R_0\,\left[\mathrm{Im}(\Delta_k^e)\,|\Psi_k^e|^{\,2} + \mathrm{Im}(\Delta_k^o)\,|\Psi_k^o|^{\,2}\right].$$

- Plasma cannot exert net torque on itself, so

$$T_\phi = \sum_{k=1,K} \delta T_k = 0.$$

Symmetry Properties of Stability Matrices

- Constraint that $T_\phi = 0$ leads to symmetry constraints on stability matrices:

$$
\begin{aligned}
E^{e}_{kk'} &= E^{e\,*}_{k'k}, \\
E^{o}_{kk'} &= E^{o\,*}_{k'k}, \\
\Gamma'_{kk'} &= \Gamma^{*}_{k'k}
\end{aligned}
$$

  for $k, k' = 1, K$.

- Extent to which these symmetries are respected is sensitive test of accuracy of numerical solution in TOMUHAWC.

Example Calculation

- For purposes of benchmarking, TOMUHAWC used to calculate $n = 1$ stability matrix for simple, circular cross-section, fixed boundary equilibrium for which $a/R_0 = 0.2$.

- Equilibrium profiles:

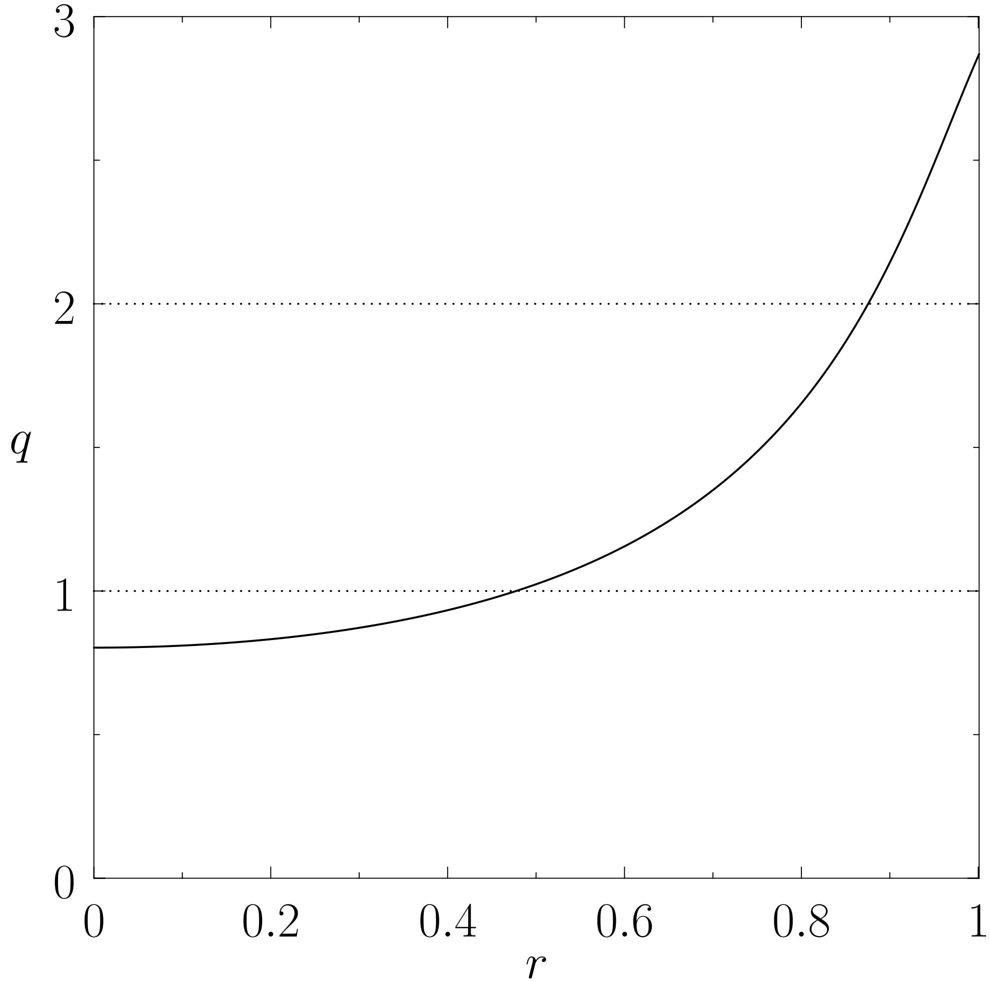
$$p(r) = p_0\,(1 - r^4)^2,$$

$$g(r) = g_0\,(1 - r^2)^{\mu} - p_0\,(1 - r^4)^2,$$

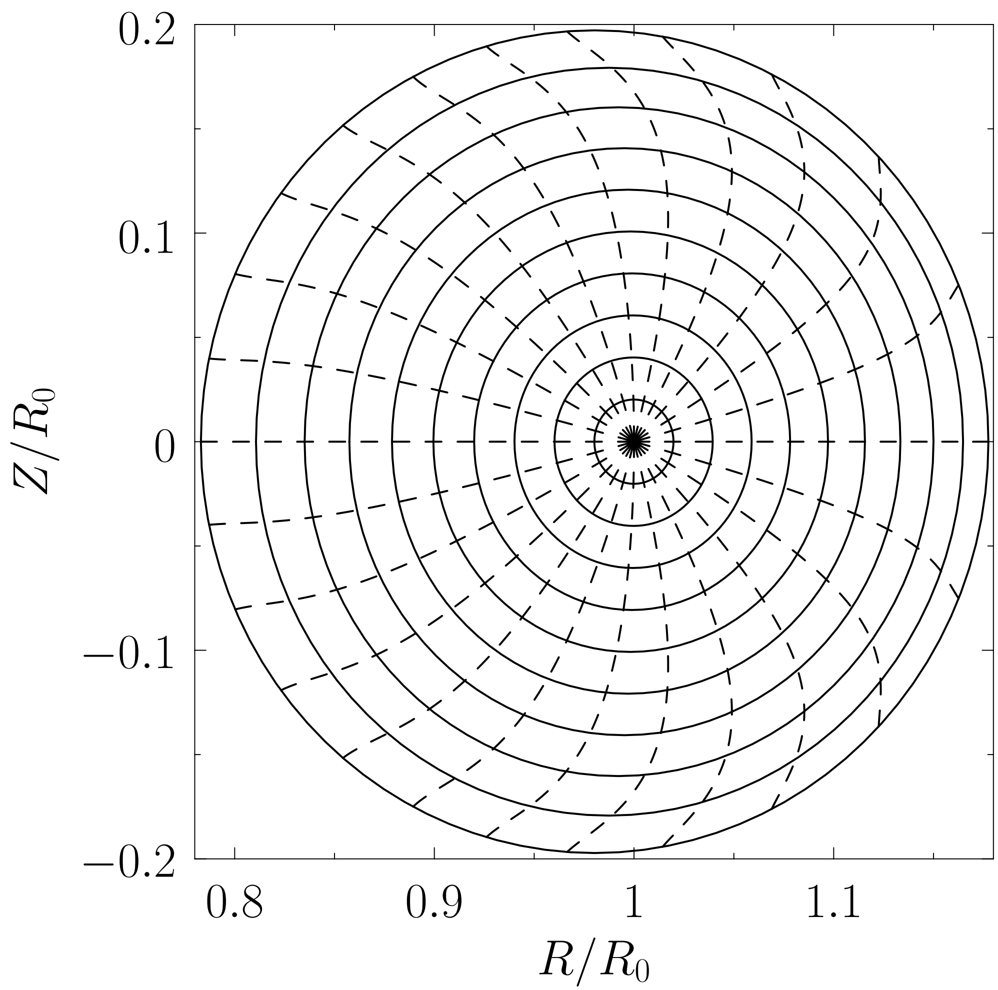
where $\mu = 3.712$, $p_0 = 1 \times 10^{-2}$, and $g_0 = 1.684 \times 10^{-2}$.

- Equilibrium parameters: $q(0) = 0.8028$, $q(1) = 2.863$, $l_i = 1.266$, $\beta = 1.096 \times 10^{-2}$, $\beta_p = 1.257$, $\beta_N = 2.708$.

Safety-Factor Profile



Flux Coordinates



Stability Matrices

$$
\mathbf{E}^{\mathrm{e}} = \begin{pmatrix} 1.7263 \times 10^{3}, & -1.9753 \times 10^{2} \\ -1.9752 \times 10^{2}, & 2.2002 \times 10^{1} \end{pmatrix},
$$

$$
\mathbf{E}^{\mathrm{o}} = \begin{pmatrix} 1.6653 \times 10^{6}, & -4.1031 \times 10^{3} \\ -4.1030 \times 10^{3}, & 1.6187 \times 10^{2} \end{pmatrix},
$$

$$
\mathbf{\Gamma} = \begin{pmatrix} -5.4889 \times 10^{4}, & 1.2676 \times 10^{2} \\ 6.9992 \times 10^{3}, & 6.8903 \times 10^{1} \end{pmatrix},
$$

$$
\mathbf{\Gamma}' = \begin{pmatrix} -5.4880 \times 10^{4}, & 6.9991 \times 10^{3} \\ 1.2674 \times 10^{2}, & 6.8903 \times 10^{1} \end{pmatrix}.
$$

- Matrices satisfy symmetry constraints to high degree of accuracy.

Dispersion Relation

- Complex growth-rate, γ, obtained by solving

$$\begin{vmatrix} E^e_{11} - \Delta^e_1(\gamma), & E^e_{12}, & \Gamma_{11}, & \Gamma_{12} \\ E^e_{12}, & E^e_{22} - \Delta^e_2(\gamma), & \Gamma_{21}, & \Gamma_{22} \\ \Gamma_{11}, & \Gamma_{21}, & E^o_{11} - \Delta^o_1(\gamma), & E^o_{12} \\ \Gamma_{12}, & \Gamma_{22}, & E^o_{21}, & E^o_{22} - \Delta^o_2(\gamma) \end{vmatrix} = 0.$$

Glasser-Greene-Johnson Resistive Layer Theory

- According to Glasser-Greene-Johnson resistive layer theory[a]

$$\Delta_k^{e,o} = S_k^{(2/3)\sqrt{D_{I\,k}}}\,\tilde{\Delta}_k^{e,o}(Q_k, E_k, F_k, G_k, H_k, K_k),$$

where S_k is Lundquist number, Q_k is normalized growth-rate, and E_k, F_k, G_k, K_k are standard GGJ parameters, evaluated at kth rational surface. Note that $\tilde{\Delta}_k^{e,o}$, Q_k, E_k, etc., independent of S_k.

- Here,

$$Q_k = (\gamma - i\,n\,\Omega_k)\,S_k^{1/3}\,\omega_{A\,k}^{-1},$$

where Ω_k and $\omega_{A\,k}$ are the toroidal angular velocity and shear-Alfvén frequency at kth rational surface.

[a] A.H. Glasser, J.M. Greene, and J.L. Johnson, Phys. Fluids **18**, 875 (1975).

Modified Dispersion Relation

- The dispersion relation becomes

$$\begin{vmatrix} E^{e}_{11} - S_1^{(2/3)\sqrt{D_{I\,1}}}\,\tilde{\Delta}^{e}_{1}, & E^{e}_{12}, & \Gamma_{11}, & \Gamma_{12} \\ E^{e}_{12}, & E^{e}_{22} - S_2^{(2/3)\sqrt{D_{I\,2}}}\,\tilde{\Delta}^{e}_{2}, & \Gamma_{21}, & \Gamma_{22} \\ \Gamma_{11}, & \Gamma_{21}, & E^{o}_{11} - S_1^{(2/3)\sqrt{D_{I\,1}}}\,\tilde{\Delta}^{o}_{1}, & E^{o}_{12} \\ \Gamma_{12}, & \Gamma_{22}, & E^{o}_{21}, & E^{o}_{22} - S_2^{(2/3)\sqrt{D_{I\,2}}}\,\tilde{\Delta}^{o}_{2} \end{vmatrix} = 0,$$

where the S dependence has now been made explicit.

Infinite-S Limit

- In the high-S limit (i.e., S_1, $S_2 \rightarrow \infty$), the dispersion relation reduces to [a]

$$\tilde{\Delta}_1^e = 0 \quad \rightarrow \quad Q_1 = -1.3054 \times 10^{-1} \pm 2.2608 \times 10^{-1}\,\mathrm{i}$$

$$\tilde{\Delta}_1^o = 0 \quad \rightarrow \quad Q_1 = -5.6846 \times 10^{-1} \pm 9.8461 \times 10^{-1}\,\mathrm{i}$$

$$\tilde{\Delta}_2^e = 0 \quad \rightarrow \quad Q_2 = -9.5738 \times 10^{-2} \pm 1.6581 \times 10^{-1}\,\mathrm{i}$$

$$\tilde{\Delta}_2^o = 0 \quad \rightarrow \quad Q_2 = -1.7984 \times 10^{-1} \pm 3.1149 \times 10^{-1}\,\mathrm{i}$$

- All modes stable due to dominant stabilizing influence of favorable average field-line curvature in tokamaks at very high S.

[a] A.H. Glasser, S.C. Jardin, and G. Tesauro, Phys. Fluids **27**, 1225 (1984).

Finite-S Limit

- At finite S, non-trivial roots of dispersion relation (i.e., roots significantly different from infinite-S roots) possible when

$$0 \leq |\tilde{\Delta}_1^e| \ll 1,$$

or

$$0 \leq |\tilde{\Delta}_1^o| \ll 1,$$

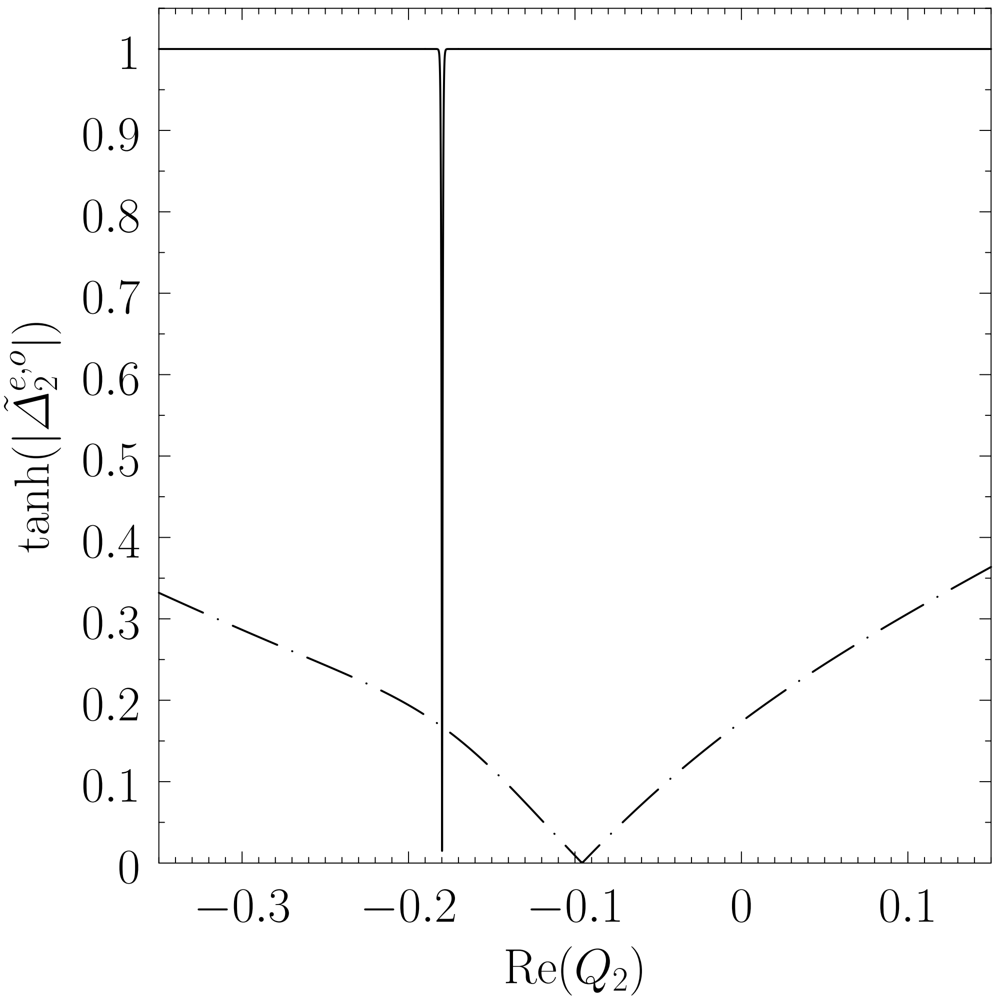
or

$$0 \leq |\tilde{\Delta}_2^e| \ll 1,$$

or

$$0 \leq |\tilde{\Delta}_2^o| \ll 1.$$

Finite-S Limit

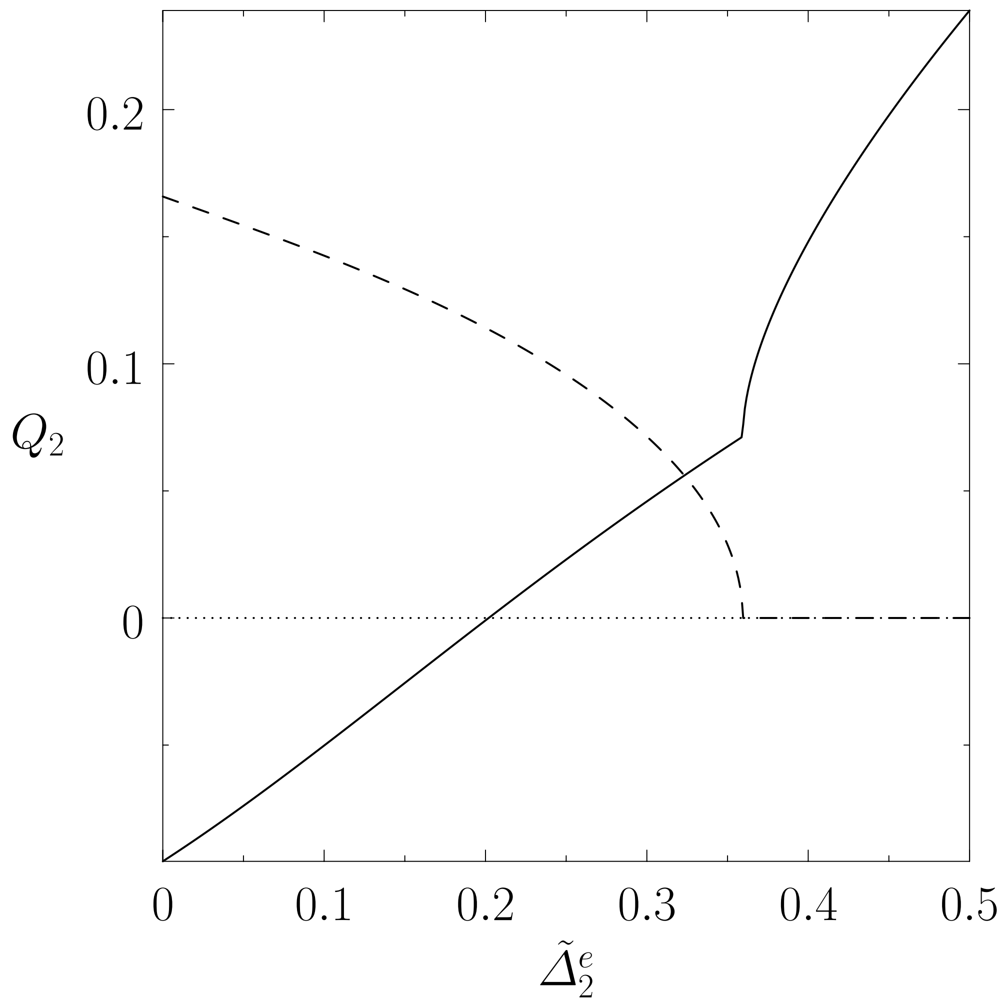


Mode Decoupling

- Twisting (interchange) parity modes decouple from one another, from tearing parity modes, and from outer region, as consequence of layer physics. These modes generally stable (at low n).
- Modest levels of sheared toroidal rotation cause tearing parity modes to decouple from one another (impossible to find common γ value that allows multiple Q_k to be simultaneously small).
- In example calculation, complicated dispersion relation simplifies considerably to give

$$
\begin{aligned}
E_{11}^e - S_1^{(2/3)\sqrt{D_{I\,1}}}\,\tilde{\Delta}_1^e &\simeq 0,\\
E_{22}^e - S_2^{(2/3)\sqrt{D_{I\,2}}}\,\tilde{\Delta}_2^e &\simeq 0.
\end{aligned}
$$

Layer Solution



Critical Lundquist Number

- Tearing parity mode that reconnects magnetic flux at $q = 1$ surface unstable when $\tilde{\Delta}_1^e > \tilde{\Delta}^e_{1\,\rm crit} = 8.975\times 10^{-3}$. So, mode unstable when

$$S_1 > S_{1\,\rm crit} = \left(E^e_{11}/\tilde{\Delta}^e_{1\,\rm crit}\right)^{(3/2)/\sqrt{D_{I\,1}}} = 7.39\times 10^{15}.$$

- Tearing parity mode that reconnects magnetic flux at $q = 2$ surface unstable when $\tilde{\Delta}_2^e > \tilde{\Delta}^e_{2\,\rm crit} = 0.2020$. So, mode unstable when

$$S_2 > S_{2\,\rm crit} = \left(E^e_{22}/\tilde{\Delta}^e_{2\,\rm crit}\right)^{(3/2)/\sqrt{D_{I\,2}}} = 7.54\times 10^{5}.$$

General Conclusions

- For small-n modes, only outer solution data that actually matters is diagonal elements of $\mathbf{E}^e$ matrix.
- According to linear GGJ layer physics, tearing parity mode that reconnects magnetic flux at given rational surface unstable when local Lunquist number exceeds critical value that depends on single parameter obtained from layer solution, and appropriate diagonal element of $\mathbf{E}^e$ matrix.
- According to nonlinear GGJ island physics, as soon as critical Lundquist number at given rational surface exceeded, appropriate tearing mode grows to large amplitude (i.e., similar to amplitude in absence of Glasser effect.) Ultimate message of resistive-MHD $\rightarrow$ tokamak plasmas intrinsically *metastable*.