

A drift-magnetohydrodynamical fluid model of helical magnetic island equilibria in the pedestals of H-mode tokamak plasmas

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A drift-magnetohydrodynamical (MHD) fluid model is developed for an isolated, steady-state, helical magnetic island chain, embedded in the pedestal of a large aspect ratio, low- β , circular cross section, H-mode tokamak plasma, to which an externally generated, multiharmonic, static magnetic perturbation whose amplitude is sufficiently large to fully relax the pedestal toroidal ion flow is applied. The model is based on a set of single helicity, reduced, drift-MHD fluid equations which take into account neoclassical poloidal and toroidal flow damping, the perturbed bootstrap current, diamagnetic flows, anomalous cross-field diffusion, average magnetic-field line curvature, and coupling to drift-acoustic waves. These equations are solved analytically in a number of different ordering regimes by means of a systematic expansion in small quantities. For the case of a freely rotating island chain, the main aims of the calculation are to determine the chain's phase velocity, and the sign and magnitude of the ion polarization term appearing in its Rutherford radial width evolution equation. For the case of a locked island chain, the main aims of the calculation are to determine the sign and magnitude of the polarization term. © 2010 American Institute of Physics. [doi:10.1063/1.3432720]

I. INTRODUCTION

The ITER tokamak¹ is designed to operate using a particular type of plasma discharge, called an *H-mode*,² which is characterized by strong density and temperature gradients localized in a (radially) thin annular region, known as the *pedestal*, which is situated just inside the last closed magnetic flux surface. Unfortunately, such gradients drive an intermittent instability known as an *edge localized mode* (ELM),³ and the large impulsive heat flux across the plasma boundary which is typically associated with this instability leads to an unacceptable limitation on the lifetime of the ITER divertor plates.⁴ Consequently, it has become essential to the success of the ITER project to find a reliable method for suppressing ELMs in H-mode tokamak plasmas.

In experiments recently performed on the DIII-D (Ref. 5) and JET (Ref. 6) tokamaks, application of an externally generated, nonaxisymmetric, static magnetic perturbation, with a broad spectrum of helical harmonics, many of which were resonant in the pedestal, to an H-mode discharge was found to either completely suppress, or greatly mitigate, the ELMs. The original motivation for these so-called *resonant magnetic perturbation* (RMP) experiments was to create a set of overlapping, static, helical magnetic island chains^{7,8} in the pedestal, thereby causing the magnetic field there to become ergodic.^{9–11} However, it appears likely that this did not actually occur (since there was no collapse in the pedestal electron temperature). Instead, magnetic island formation was (presumably) suppressed to a large extent by equilibrium plasma flows,^{12–19} and only a few nonoverlapping island chains were produced. The purpose of this paper is to investigate the physics of such chains.

For a number of reasons, the physics of an isolated helical island chain generated in the pedestal of an H-mode

tokamak plasma, during an RMP experiment, is significantly different from that of a conventional island chain: e.g., a chain generated by a *neoclassical tearing mode*²⁰ (NTM) resonant in the plasma core. First, the pedestal has *much smaller* density (and temperature) scale length than the core plasma. Second, an RMP induced island chain is necessarily *nonrotating*, since it is locked to that helical harmonic of the externally generated, *static* magnetic perturbation which *resonates* at its associated rational surface.²¹ An NTM island chain, on the other hand, is convected by equilibrium ion flows in the plasma core, and therefore rotates.²² Finally, the *nonresonant* harmonics of the external magnetic perturbation in an RMP experiment generally produce a significant *toroidal flow damping* effect²³ which relaxes the toroidal ion flow in the pedestal toward a fixed value determined by neoclassical theory.^{24–26} (The resonant harmonic of the external perturbation, as well as the magnetic perturbation associated with the island itself, also contribute to the toroidal flow damping.²⁷) Of course, the poloidal ion flow is already relaxed to a fixed value determined by neoclassical theory,^{28,29} due to the strong *poloidal flow damping* which is present in all tokamak plasmas (because of the significant toroidicity-induced poloidal variation of the toroidal magnetic-field strength around magnetic flux surfaces³⁰). It follows that, in an RMP experiment with sufficiently large toroidal flow damping, the poloidal and toroidal ion flow velocities are both constrained to take *fixed values* in the pedestal. By contrast, whereas the poloidal ion flow in the core of a conventional tokamak plasma (i.e., a plasma with no significant central toroidal flow damping) is fixed, the toroidal flow is generally free to vary.

This paper investigates the equilibrium of an isolated (radially) thin, helical magnetic island chain, embedded in

the pedestal of a large aspect ratio, low- β , circular cross section, H-mode tokamak plasma, in the presence of an externally generated, nonaxisymmetric, static magnetic perturbation whose amplitude is sufficiently large to fully relax the pedestal toroidal ion flow. For the sake of completeness, we shall consider rotating island chains, in addition to chains that are locked to the resonant harmonic of the perturbation. For the case of a rotating island chain, the primary aims of the investigation are to predict the chain's steady-state *phase velocity*, and the magnitude and sign of the *ion polarization current* term appearing in its Rutherford radial width evolution equation.³¹ For the case of a locked island chain, whose phase velocity is necessarily zero, the primary aim is to predict the magnitude and sign of the polarization term. In both cases, the starting point for the investigation is a set of steady-state, single helicity, reduced,³² drift-magneto-hydrodynamical (MHD),³³ *fluid equations*.^{22,34} These equations incorporate *neoclassical* effects such as poloidal and toroidal flow damping, the perturbed bootstrap current,³⁵ and enhanced ion inertia,^{36,37} as well as more conventional effects such as shear-Alfvén waves, diamagnetic flows (including the modification of ion diamagnetism due to the gyroviscosity tensor³³), resistivity, anomalous cross-field diffusion of particles and momentum, *average* magnetic-field line curvature,³⁸ and coupling to drift-acoustic waves.³⁹ On the other hand, the equations neglect compressional-Alfvén waves, kinetic effects,^{40–42} ion orbit widths (relative to the radial island width),⁴³ temperature gradients, electron inertia, and poloidal coupling due to toroidicity, geodesic magnetic-field line curvature,⁴⁴ or parallel viscosity. Note that the curvature, flow damping, and anomalous diffusion terms appearing in the equations are *phenomenological* in nature, and are not exact. The fundamental fluid equations are solved *analytically* in a number of distinct ordering regimes by means of a *systematic* expansion in small quantities. The regimes in question are chosen to illustrate the effect of various different levels of poloidal flow damping on both freely rotating and locked island chains. However, the regimes are also designed to ensure that the constant- ψ approximation⁴⁵ holds, and that the ion fluid is largely constrained to flow around perturbed magnetic flux surfaces, since this leads to a tremendous simplification in the analysis. (The physics of nonconstant- ψ island chains is examined in Ref. 46. Moreover, the complications which arise when an island chain becomes sufficiently narrow that the ion fluid is *not* tied to magnetic flux surfaces are discussed in Refs. 47–49.)

II. PRELIMINARY ANALYSIS

A. Fundamental definitions

Consider a large aspect ratio, low- β , circular cross section, H-mode tokamak plasma of major radius R_0 , and toroidal magnetic-field strength B_0 . Let us adopt a conventional, right-handed, quasicylindrical, toroidal coordinate system, (r, θ, φ) , whose symmetry axis ($r=0$) coincides with the magnetic axis. The coordinate r also serves as a label for the unperturbed (by the island chain) magnetic flux surfaces. Let the equilibrium toroidal magnetic field and toroidal plasma current both run in the $+\varphi$ direction. Suppose that a helical

magnetic island chain, with m_θ poloidal periods, and n_φ toroidal periods, is embedded in the pedestal of the aforementioned plasma. The island chain is assumed to be *radially localized* in the vicinity of its associated *rational surface*, minor radius r_s , which is defined as the unperturbed flux surface where $q(r_s)=m_\theta/n_\varphi\equiv q_s$. Here, $q(r)$ is the safety-factor profile. Let the full radial width of the island chain's magnetic separatrix be $4w$. Furthermore, let the chain rotate (in the laboratory frame) at the angular velocity ω . In the following, it is assumed that $\epsilon_s\equiv r_s/R_0\ll 1$ and $w/r_s\ll 1$.

It is helpful to define the *magnetic shear length*, $L_s\equiv R_0q_s/(d\ln q/d\ln r)_{r_s}$, the *density scale length*, $L_n\equiv -r_s/(d\ln n/d\ln r)_{r_s}$, and the phenomenological *mean radius of curvature of magnetic-field lines*, L_c . (Incidentally, the mean curvature is assumed to be favorable.³⁸) Here, $n(r)$ is the electron number density profile. It is also helpful to define the *ion diamagnetic velocity*, $V_{*i}\equiv T_i/(eB_0L_n)$, the *ion beta*, $\beta_i\equiv \mu_0 n_0 T_i/B_0^2$, and the *ion gyroradius*, $\rho_i\equiv (T_i/m_i)^{1/2}/(eB_0/m_i)$, where $n_0\equiv n(r_s)$, T_i is the (uniform) ion temperature, e the magnitude of the electron charge (as well as the ion charge), and m_i the ion mass. All of the aforementioned parameters are evaluated at the rational surface.

B. Fundamental parameters

The key parameters in our model are the *temperature ratio parameter*, $\tau\equiv T_e/T_i$, the *aspect-ratio parameter*, $\epsilon\equiv(\epsilon_s/q_s)^2$, the *ion gyroradius parameter*, $\rho\equiv(q_s/\epsilon_s)^2(\rho_i/w)^2$, the *density gradient parameter*, $\alpha_n\equiv(\epsilon_s/q_s)\times(L_n/L_s)(w/\rho_i)^2$, the *magnetic curvature parameter*, $\alpha_c\equiv 2(\epsilon_s/q_s)^2(L_n/L_c)(w/\rho_i)^2$, the *plasma pressure parameter*, $\beta\equiv\beta_i(q_s/\epsilon_s)^2(L_s/L_n)^2(\rho_i/w)^2$, and the *phase velocity parameter*, $v\equiv(V_p-V_p^{nc})/V_{*i}$. Here, T_e is the (uniform) electron temperature in the vicinity of the rational surface, $V_p\equiv\omega/k_\theta$ the phase velocity of the island chain in the laboratory frame, and $k_\theta\equiv m_\theta/r_s$. Note that if V_p is *positive* then the chain propagates in the *electron* diamagnetic direction, and vice versa. Furthermore, $V_p^{nc}\equiv V_{\theta i}^{nc}-(\epsilon_s/q_s)V_{\varphi i}^{nc}$, where $V_{\theta i}^{nc}$ is the fixed velocity toward which neoclassical flow damping relaxes the ion poloidal velocity profile in the vicinity of the rational surface, and $V_{\varphi i}^{nc}$ is the corresponding fixed toroidal velocity. Any radial variation in these velocities across the island region is neglected. The parameter V_p^{nc} is known as the *neoclassical phase velocity*,²² and is the fixed value toward which neoclassical poloidal and toroidal flow damping relax the phase velocity of the island chain. Note that when v takes the values 0, 1, and $1+\tau$ then the island chain is effectively convected by the unperturbed local ion, $\mathbf{E}\times\mathbf{B}$, and electron fluid, respectively, at the rational surface.

The three parameters which control flow damping in our model are $\hat{\nu}_{\theta i}\equiv(\epsilon_s/q_s)^2(\nu_{\theta i}/\omega_{*i})$, $\hat{\nu}_{\varphi i}\equiv(\nu_{\varphi i}/\omega_{*i})$, and $\hat{\nu}_{\theta e}\equiv(m_e/m_i)(\epsilon_s/q_s)^2(\nu_{\theta e}/\omega_{*i})$. Here, $\omega_{*i}=k_\theta V_{*i}$ is the *ion diamagnetic frequency*, $\nu_{\theta i}$ the phenomenological *ion poloidal flow damping rate*, $\nu_{\varphi i}$ the phenomenological *ion toroidal flow damping rate* (due to the combined effect of the resonant and nonresonant harmonics of the external magnetic perturbation), $\nu_{\theta e}$ the phenomenological *electron poloidal*

flow damping rate, and m_e the electron mass. All quantities are evaluated at the rational surface. Any radial variation in the damping rates across the island region is neglected.

Finally, the three parameters which control cross flux-surface transport in our model are $\eta \equiv \eta_{\parallel}/(\mu_0 \omega_{*i} w^2)$, $D \equiv D_{\perp}/(\omega_{*i} w^2)$, and $\mu \equiv \mu_{\perp i}/(n_0 m_i \omega_{*i} w^2)$. Here, η_{\parallel} is the parallel electrical resistivity, D_{\perp} the phenomenological perpendicular particle diffusivity (due to small scale plasma turbulence), and $\mu_{\perp i}$ the phenomenological perpendicular ion viscosity (likewise, due to small scale plasma turbulence). Again, all quantities are evaluated at the rational surface, and any radial variation in the transport coefficients across the island region is neglected.

C. Fundamental fields

Making use of a standard *single helicity* approximation, all fields in our model are assumed to depend only on the normalized radial coordinate $X \equiv (r - r_s)/w$ and the helical angle $\zeta \equiv m\theta - n\varphi - \omega t$.

The steady-state fluid equations which constitute the starting point for our investigation involve the following four fields:⁵⁰

$$\psi(X, \zeta) \equiv \left(\frac{L_s}{B_0 w^2} \right) A_{\parallel}, \quad (1)$$

$$N(X, \zeta) \equiv \left(\frac{L_n}{w} \right) \left(\frac{n - n_0}{n_0} \right), \quad (2)$$

$$\phi(X, \zeta) \equiv - \left(\frac{\Phi}{w V_{*i} B_0} \right) + \left(\frac{V_p}{V_{*i}} X \right), \quad (3)$$

$$V(X, \zeta) \equiv \left(\frac{\epsilon_s}{q_s} \right) \left(\frac{V_{\parallel i} - V_{\parallel i}^{nc}}{V_{*i}} \right), \quad (4)$$

as well as the auxiliary field

$$J(X, \zeta) \equiv \beta^{-1} \left(-1 + \frac{L_s}{B_0} \nabla^2 A_{\parallel} \right). \quad (5)$$

Here, A_{\parallel} is the parallel (to the equilibrium magnetic field at the rational surface) magnetic vector potential, Φ the scalar electric potential, and $V_{\parallel i}$ the parallel ion fluid velocity. The field ψ serves as a label for the perturbed magnetic flux surfaces; the field N measures the chain-induced modification to the electron number density profile; the field ϕ is a stream function for the $\mathbf{E} \times \mathbf{B}$ flow in a frame of reference which co-rotates with the island chain; the field V parameterizes the chain-induced deviation of the ion parallel velocity profile from its fixed neoclassical value; and the field J measures the perturbed parallel current density.

D. Fundamental fluid equations

Our fundamental set of steady-state, single helicity, reduced, drift-MHD, fluid equations takes the form^{22,34}

$$0 = [\phi + \tau N, \psi] + \beta \eta J + \alpha_n^{-1} \hat{v}_{\theta e} \times \{ \alpha_n^{-1} J + V - \partial_X(\phi + \tau N) + v - 1 - \tau \}, \quad (6)$$

$$0 = [\phi, N] - \rho [\alpha_n V + J, \psi] - \alpha_c \rho [\phi + \tau N, X] + D \partial_X^2 N - \rho \hat{v}_{\theta e} \{ \alpha_n^{-1} J + V - \partial_X(\phi + \tau N) + v - 1 - \tau \}, \quad (7)$$

$$0 = \epsilon \partial_X [\phi - N, \partial_X \phi] + [J, \psi] + (1 + \tau) \alpha_c [N, X] + \epsilon \mu \partial_X^4 (\phi - N) + \hat{v}_{\theta i} \partial_X \{ V - \partial_X(\phi - N) + v \} - \epsilon \hat{v}_{\phi i} \partial_X V + \hat{v}_{\theta e} \partial_X \{ \alpha_n^{-1} J + V - \partial_X(\phi + \tau N) + v - 1 - \tau \}, \quad (8)$$

$$0 = [\phi, V] - (1 + \tau) \alpha_n [N, \psi] + \mu \partial_X^2 V - \hat{v}_{\theta i} \times \{ V - \partial_X(\phi - N) + v \} - \hat{v}_{\phi i} V - \hat{v}_{\theta e} \times \{ \alpha_n^{-1} J + V - \partial_X(\phi + \tau N) + v - 1 - \tau \}, \quad (9)$$

$$0 = \partial_X^2 \psi - 1 - \beta J. \quad (10)$$

Here, $\partial_X \equiv \partial / \partial X|_{\zeta}$, $\partial_{\zeta} \equiv \partial / \partial \zeta|_X$, and $[A, B] \equiv \partial_X A \partial_{\zeta} B - \partial_{\zeta} A \partial_X B$. Equation (6) is the parallel component of Ohm's law, Eq. (7) is the parallel component of the curl of Ohm's law (combined with the continuity equation), Eq. (8) is the parallel component of the plasma vorticity equation, and Eq. (9) is the parallel component of the plasma equation of motion. Finally, Eq. (10) is obtained directly from Maxwell's equations. Equations (8) and (9) can be combined to give

$$0 = [J, \psi] - (1 + \tau) \alpha_c [N, X] + \partial_X G, \quad (11)$$

where

$$G \equiv \epsilon [\phi - N, \partial_X \phi] + [\phi, V] - (1 + \tau) \alpha_n [N, \psi] + \mu \partial_X^2 \{ V + \epsilon \partial_X(\phi - N) \} - (1 + \epsilon) \hat{v}_{\phi i} V. \quad (12)$$

Equations (6)–(11) are subject to the boundary conditions

$$\psi(X, \zeta) \rightarrow \frac{1}{2} X^2 + \cos \zeta, \quad (13)$$

$$N(X, \zeta) \rightarrow -X, \quad (14)$$

$$\phi(X, \zeta) \rightarrow -(1 - v)X, \quad (15)$$

$$V(X, \zeta) \rightarrow 0, \quad (16)$$

$$J(X, \zeta) \rightarrow 0, \quad (17)$$

as $|X| \rightarrow \infty$. Equations (13), (14), and (17) are obtained via asymptotic matching to the perturbed plasma equilibrium, assuming that the island chain is radially thin (i.e., $w \ll r_s$). Furthermore, Eqs. (15) and (16) ensure that the ion poloidal and toroidal velocities both asymptote to their fixed neoclassical values far from the chain. This follows because

$$\frac{V_{\theta i} - V_{\theta i}^{nc}}{V_{*i}} = V - \partial_X(\phi - N) + v, \quad (18)$$

$$\frac{V_{\phi i} - V_{\phi i}^{nc}}{(q_s / \epsilon_s) V_{*i}} = V, \quad (19)$$

where $V_{\theta i}(X, \zeta)$ and $V_{\phi i}(X, \zeta)$ are the respective velocities.

E. Determination of island chain radial width

Standard asymptotic matching^{21,31,51} reveals that the so-called ‘‘Rutherford’’ equation which determines the island chain’s radial width takes the form

$$\begin{aligned} \frac{dw}{dt} \propto & \Delta' r_s + 2m_\theta \left(\frac{w_v}{w} \right)^2 \cos \phi \\ & + J_c \beta_i \left(\frac{q_s}{\epsilon_s} \right)^2 \left(\frac{L_s}{L_n} \right)^2 \left(\frac{\rho_i}{r_s} \right)^2 \left(\frac{r_s}{w} \right)^3, \end{aligned} \quad (20)$$

where Δ' is the tearing stability index for the m_θ, n_φ mode,⁴⁵ $4w_v$ the full radial width of the vacuum island chain associated with the *resonant* harmonic of the static external magnetic perturbation, ϕ the helical phase difference between the island chain and the vacuum island chain, and

$$J_c \equiv -2 \int_{-\infty}^{\infty} \oint J \cos \zeta dX \frac{d\zeta}{2\pi}. \quad (21)$$

In deriving Eq. (20), we have made the simplifying assumption that the equilibrium plasma current external to the rational surface is negligible. Of course, a steady-state island chain is characterized by $dw/dt=0$. The first term on the right-hand side of Eq. (20) parameterizes the intrinsic MHD stability of the island chain, the second measures the destabilizing effect of the resonant harmonic of the external perturbation, and the third parameterizes the combined stabilizing or destabilizing effect of the *ion polarization current*, *mean magnetic-field line curvature*, and the *perturbed bootstrap current*. Note, incidentally that Δ' becomes a function of w for a sufficiently wide island chain.^{52–54}

F. Determination of island chain helical phase

Standard asymptotic matching^{21,31,51} also shows that the helical phase of the island chain evolves as

$$\begin{aligned} \frac{d^2\phi}{dt^2} \propto & -2m_\theta \left(\frac{w_v}{r_s} \right)^2 \left(\frac{w}{r_s} \right)^2 \sin \phi \\ & + J_s \beta_i \left(\frac{q_s}{\epsilon_s} \right)^2 \left(\frac{L_s}{L_n} \right)^2 \left(\frac{\rho_i}{r_s} \right)^2 \left(\frac{w}{r_s} \right), \end{aligned} \quad (22)$$

where

$$J_s = -2 \int_{-\infty}^{\infty} \oint J \sin \zeta dX \frac{d\zeta}{2\pi}. \quad (23)$$

Of course, a steady-state island chain is characterized by $d^2\phi/dt^2=0$. The first term on the right-hand side of Eq. (22) parameterizes the *electromagnetic torque* exerted on the plasma in the vicinity of the island chain by the *resonant* harmonic of the external perturbation,^{21,51} and the second term parameterizes the *neoclassical drag torque* due to equilibrium ion flow relative to the chain.^{22,55}

In the limit in which the resonant electromagnetic torque

is too weak to significantly affect the chain’s helical phase (i.e., $w_v \rightarrow 0$), the steady-state version of Eq. (22) simply reduces to

$$J_s = 0. \quad (24)$$

The above criterion determines the phase velocity parameter for a *freely rotating* island chain (i.e., a chain which is not subject to an externally generated, resonant electromagnetic torque). On the other hand, if the resonant electromagnetic torque is sufficiently large to *lock* the chain to the static external perturbation then the chain’s phase velocity is necessarily *zero*, and its phase velocity parameter takes the fixed value

$$v_l \equiv -\frac{V_{*i}^{nc}}{V_{*i}}. \quad (25)$$

G. Fundamental expansion procedure

Equations (6)–(11) are solved via an expansion in *two* small parameters, Δ and δ , where $\Delta \ll \delta \ll 1$.⁵⁶ The expansion procedure is as follows. First, the coordinates X and ζ are assumed to be $\mathcal{O}(\Delta^0 \delta^0)$. Next, some particular ordering scheme is adopted for the thirteen physics parameters $v, \tau, \epsilon, \rho, \alpha_n, \alpha_c, \beta, \hat{v}_{\theta i}, \hat{v}_{\varphi i}, \hat{v}_{\theta e}, \eta, D$, and μ . The fields ψ, N, ϕ, V , and J are then expanded in the form $\psi(X, \zeta) = \sum_{i,j=0,\infty} \psi_{i,j}(X, \zeta)$, etc., where $\psi_{i,j} \sim \mathcal{O}(\Delta^i \delta^j)$. Finally, Eqs. (6)–(11) are solved order by order. This procedure is applied to *three* distinct ordering schemes that are designed to illustrate the effect of various different levels of ion poloidal flow damping on both freely rotating and locked island chains.

III. STRONG POLOIDAL FLOW DAMPING REGIME

A. Ordering scheme

The ordering scheme adopted in the so-called *strong poloidal flow damping regime* is

$$\Delta^0 \delta^{-1}: \hat{v}_{\theta i},$$

$$\Delta^0 \delta^0: v, \tau, \alpha_n, \alpha_c,$$

$$\Delta^0 \delta^1: \epsilon, \rho, \beta,$$

$$\Delta^1 \delta^0: \eta, D, \mu, \hat{v}_{\varphi i},$$

$$\Delta^1 \delta^1: \hat{v}_{\theta e}.$$

This scheme implies that

$$v_{\theta i} \gg (\epsilon_s/q_s)^2 v_{\theta i} \gg \omega_{*i} \gg v_\eta v_D, v_\mu, v_{\varphi i}, (m_e/m_i) v_{\theta e}, \quad (26)$$

where $v_\eta = \eta_{\parallel} / \mu_0 \omega^2$, $v_D = D_{\perp} / w^2$, and $v_\mu = \mu_{\perp i} / n_0 m_i w^2$ are the radial diffusion rates for magnetic flux, particles, and ion momentum, respectively, across the island region. According to the above inequality, the ion poloidal flow damping term is *dominant* in both the parallel plasma vorticity equation (8), and the parallel plasma equation of motion, Eq. (9). Furthermore, the perpendicular diffusion terms, as well as the ion

toroidal and electron poloidal flow damping terms, are all *very small* compared to the leading order terms in Eqs. (6)–(11). (This is a natural assumption for the diffusion terms, since $\omega_{*i} \sim \nu_{\eta} \nu_D$, and ν_{μ} is a characteristic ordering for a comparatively narrow linear layer, rather than a comparatively wide nonlinear island chain, and $\nu_{\eta} \nu_D$, and $\nu_{\mu} \propto w^{-2}$.) The above ordering scheme also implies that

$$|V_p - V_p^{nc}| \sim V_{*i}, \quad (27)$$

$$L_n \ll \left(\frac{\epsilon_s}{q_s}\right) L_s, \quad (28)$$

$$\left(\frac{q_s}{\epsilon_s}\right) \rho_i \ll w \ll \left(\frac{L_s}{L_n}\right) \rho_i, \quad (29)$$

$$\beta_i \ll \left(\frac{\epsilon_s}{q_s}\right)^2, \quad (30)$$

$$L_c \sim \left(\frac{\epsilon_s}{q_s}\right) L_s. \quad (31)$$

Note that the inequality (28) is only likely to hold in the pedestal of an H-mode tokamak plasma, where the density scale length is relatively small [typically, $L_n \sim (\epsilon_s/q_s)L_s$ in the core of such a plasma]. The inequality (29) implies that the island chain is much wider than the poloidal ion gyroradius, $\rho_{\theta i} \equiv (q_s/\epsilon_s)\rho_i$ (and is, thus, also much wider than a typical trapped ion orbit), but still too narrow for ion acoustic waves to play an important role in the island dynamics [since such waves are generally only important when $w \approx (L_s/L_n)\rho_i$ (Ref. 57)]. Finally, inequality (30) ensures that the ion polarization current in the vicinity of the island chain does not become large enough to invalidate the constant- ψ approximation.^{45,58}

B. Zeroth order solution

To zeroth order in the primary and secondary expansions (i.e., to order $\Delta^0 \delta^0$), Eqs. (6), (7), (11), (9), and (10) yield

$$0 = [\phi_{0,0} + \tau N_{0,0}, \psi_{0,0}], \quad (32)$$

$$0 = [\phi_{0,0}, N_{0,0}], \quad (33)$$

$$0 = [J_{0,0}, \psi_{0,0}] + (1 + \tau)\alpha_c [N_{0,0}, X] + \partial_X \{[\phi_{0,0}, V_{0,0}] - (1 + \tau)\alpha_n [N_{0,0}, \psi_{0,0}]\}, \quad (34)$$

$$0 = -\hat{\nu}_{\theta i} \{V_{0,0} - \partial_X(\phi_{0,0} - N_{0,0}) + v\}, \quad (35)$$

$$0 = \partial_X^2 \psi_{0,0} - 1, \quad (36)$$

respectively.

Equations (13) and (36) give

$$\psi_{0,0} = \frac{1}{2}X^2 + \cos \zeta \equiv \Omega(X, \zeta). \quad (37)$$

Hence, the lowest order magnetic flux-function, $\Omega(X, \zeta)$, maps out a constant- ψ magnetic island chain whose O-points lie at $X=0$, $\zeta=\pi$, and $\Omega=-1$, whose X-points lie at $X=0$, $\zeta=0$, and $\Omega=1$, and whose magnetic separatrix corresponds to $\Omega=1$.³¹

Equations (14), (15), (32), and (33) yield

$$\phi_{0,0} = s\phi_0(\Omega), \quad (38)$$

$$N_{0,0} = sN_0(\Omega), \quad (39)$$

where $s \equiv \text{sgn}(X)$. It follows that the lowest order electrostatic potential (in the frame of the island chain) and density profiles are *odd* (in X) *magnetic flux-surface functions*. This implies that, to lowest order, the ion and electron fluids [whose stream functions (in the island frame) are $\phi-N$ and $\phi+\tau N$, respectively] do not cross magnetic flux surfaces. Let

$$M(\Omega) \equiv -\frac{d\phi_0}{d\Omega}, \quad (40)$$

$$L(\Omega) \equiv -\frac{dN_0}{d\Omega}. \quad (41)$$

Note that $\phi_0=N_0=M=L=0$ within the magnetic separatrix (i.e., $-1 \leq \Omega < 1$), since it is impossible to have an odd flux-surface function in this region. This means that the density profile is *locally flattened* by the island chain.⁵⁷ Moreover, the boundary conditions (14) and (15) reduce to

$$M(\Omega \rightarrow \infty) \rightarrow \frac{1-v}{\sqrt{2\Omega}}, \quad (42)$$

$$L(\Omega \rightarrow \infty) \rightarrow \frac{1}{\sqrt{2\Omega}}. \quad (43)$$

Note that the flattening of the density profile is *not* due to the action of ion acoustic waves. Instead, the flattening is attributable to the fact that, to lowest order, the density is simply convected by the $\mathbf{E} \times \mathbf{B}$ flow [i.e., $[\phi, N]$ is the dominant term in Eq. (7)], combined with the fact that the lowest order electron stream function, $\phi + \tau N$, is constrained to be a flux-surface function [since $[\phi + \tau N, \psi]$ is the dominant term in Eq. (6)].

Equation (35) gives

$$V_{0,0} = \partial_X(\phi_{0,0} - N_{0,0}) - v = -|X|(M - L) - v. \quad (44)$$

Observe that this expression automatically satisfies the boundary condition (16).

Finally, Eq. (34) yields

$$\begin{aligned}
 [J_{0,0}, \psi_{0,0}] &= -(1 + \tau)\alpha_c [N_{0,0}, X] \\
 &\quad - \partial_X \{ [\phi_{0,0}, V_{0,0}] - (1 + \tau)\alpha_n [N_{0,0}, \psi_{0,0}] \},
 \end{aligned} \tag{45}$$

which reduces to

$$[J_{0,0}, \Omega] = [(1/2)d_\Omega [M(M - L)]X^2 - (1 + \tau)\alpha_c L|X|, \Omega], \tag{46}$$

where $d_\Omega \equiv d/d\Omega$. It follows that

$$J_{0,0} = (1/2)d_\Omega [M(M - L)]\widetilde{X^2} - (1 + \tau)\alpha_c L|\widetilde{X}| + \bar{J}(\Omega).$$

Here,

$$\bar{A} \equiv A - \langle A \rangle / \langle 1 \rangle, \tag{48}$$

and

$$\langle A(s, \Omega, \zeta) \rangle \equiv \begin{cases} \oint \frac{A(s, \Omega, \zeta)}{[2(\Omega - \cos \zeta)]^{1/2}} \frac{d\zeta}{2\pi} & 1 \leq \Omega \\ \oint_{\zeta_0}^{2\pi - \zeta_0} \frac{A(s, \Omega, \zeta) + A(-s, \Omega, \zeta)}{2[2(\Omega - \cos \zeta)]^{1/2}} \frac{d\zeta}{2\pi} & -1 \leq \Omega < 1, \end{cases} \tag{49}$$

where $\zeta_0 = \cos^{-1}(\Omega)$ (with $0 \leq \zeta_0 \leq \pi$). The operator $\langle \dots \rangle$ is the lowest order *magnetic flux-surface average*. Note that $\langle \bar{A} \rangle \equiv 0$ and $\langle [A, \Omega] \rangle \equiv 0$ for any $A(X, \zeta)$.

The expression (47) for the lowest order perturbed parallel current density, $J_{0,0}(\Omega, \zeta)$, contains three unknown flux-surface functions, $M(\Omega)$, $L(\Omega)$, and $\bar{J}(\Omega)$. Moreover, although $J_{0,0}$ contributes to the *cosine integral* (21), it makes no contribution to the *sine integral* (23) (since $J_{0,0}$ has the symmetry of $\cos \zeta$). In order to determine the three unknown flux-surface functions, as well as the value of the sine integral, it is necessary to go to higher order in our primary expansion scheme.

C. First order solution

To first order in the primary expansion, and zeroth order in the secondary expansion (i.e., to order $\Delta^1 \delta^0$), Eqs. (6), (7), (11), (9), and (10) yield

$$0 = [\phi_{1,0} + \tau N_{1,0}, \psi_{0,0}] + [\phi_{0,0} + \tau N_{0,0}, \psi_{1,0}], \tag{50}$$

$$0 = [\phi_{1,0}, N_{0,0}] + [\phi_{0,0}, N_{1,0}] + D\partial_X^2 N_{0,0}, \tag{51}$$

$$0 = [J_{1,0}, \psi_{0,0}] + [J_{0,0}, \psi_{1,0}] + (1 + \tau)\alpha_c [N_{1,0}, X] + \partial_X G, \tag{52}$$

$$0 = -\hat{\nu}_{\theta i} \{ V_{1,0} - \partial_X (\phi_{1,0} - N_{1,0}) \}, \tag{53}$$

$$0 = \partial_X^2 \psi_{1,0}, \tag{54}$$

respectively, where

$$\begin{aligned}
 G &= [\phi_{1,0}, V_{0,0}] + [\phi_{0,0}, V_{1,0}] \\
 &\quad - (1 + \tau)\alpha_n [N_{1,0}, \psi_{0,0}] - (1 + \tau)\alpha_n [N_{0,0}, \psi_{1,0}] \\
 &\quad + \mu \partial_X^2 V_{0,0} - \hat{\nu}_{\phi i} V_{0,0}.
 \end{aligned} \tag{55}$$

Moreover, Eq. (54), (50), (53), (51), and (52) [in combination with Eqs. (38)–(41) and Eq. (44)] give

$$\psi_{1,0} = 0, \tag{56}$$

$$\phi_{1,0} = -\tau N_{1,0}, \tag{57}$$

$$V_{1,0} = -(1 + \tau)\partial_X N_{1,0}, \tag{58}$$

$$[N_{1,0}, \Omega] = \frac{D(X^2 d_\Omega L + L)}{M + \tau L}, \tag{59}$$

$$[J_{1,0}, \Omega] = -(1 + \tau)\alpha_c [N_{1,0}, X] - \partial_X G, \tag{60}$$

respectively, where

$$\begin{aligned}
 G &= \tau [N_{1,0}, |X|(M - L)] \\
 &\quad - (1 + \tau)M [|X| \partial_\Omega N_{1,0}, \Omega] - (1 + \tau)\alpha_n [N_{1,0}, \Omega] \\
 &\quad - \mu \{ |X|^3 d_\Omega^2 (M - L) + 3|X| d_\Omega (M - L) \} \\
 &\quad + \hat{\nu}_{\phi i} \{ |X|(M - L) + v \},
 \end{aligned} \tag{61}$$

and $\partial_\Omega \equiv \partial / \partial \Omega|_\zeta$. In addition, the lowest order flux-surface average of Eq. (6) reduces to

$$(\beta \eta + \alpha_n^{-2} \hat{\nu}_{\theta e}) \bar{J} \langle 1 \rangle = -(1 + \tau)\alpha_n^{-1} \hat{\nu}_{\theta e} (L - \langle 1 \rangle). \tag{62}$$

Equation (62) implies that

$$\bar{J}(\Omega) = (1 + \tau)\alpha_n \left(\frac{\epsilon \nu_{\theta e} \tau_e}{1 + \epsilon \nu_{\theta e} \tau_e} \right) (1 - L / \langle 1 \rangle), \tag{63}$$

where the *electron collision time*, τ_e , is defined $\tau_e \equiv m_e / (n_0 e^2 \eta)$.

The flux-surface average of Eq. (59), combined with the boundary condition (43), yields

$$L(\Omega) = \frac{1}{\langle X^2 \rangle}, \quad (64)$$

outside the magnetic separatrix (i.e., $\Omega > 1$). Equation (59) also gives

$$[N_{1,0}, \Omega] = \frac{D\widetilde{X^2}d_\Omega L}{M + \tau L}, \quad (65)$$

outside the separatrix. However, it is assumed that $[N_{1,0}, \Omega] = 0$ inside the separatrix.

The flux-surface average of Eq. (60) yields

$$0 = d_\Omega \{ \langle |X|G \rangle + (1 + \tau)\alpha_c \langle |X|[N_{1,0}, \Omega] \rangle \}, \quad (66)$$

outside the separatrix, whereas the flux-surface average of X times this equation gives

$$\langle X[J_{1,0}, \Omega] \rangle = -d_\Omega \{ \langle X^2G \rangle + (1/2)(1 + \tau)\alpha_c \langle X^2[N_{1,0}, \Omega] \rangle + \langle G \rangle \}. \quad (67)$$

Now, it can be demonstrated that

$$\begin{aligned} \langle |X|^j G \rangle &= (1 + j)^{-1} \{ \tau(M - L) - j(1 + \tau)M \} \\ &\quad \times d_\Omega \langle |X|^{j+1} [N_{1,0}, \Omega] \rangle + \tau d_\Omega (M - L) \\ &\quad \times \langle |X|^{j+1} [N_{1,0}, \Omega] \rangle + j(1 + \tau)M \langle |X|^{j-1} [N_{1,0}, \Omega] \rangle \\ &\quad - (1 + \tau)\alpha_n \langle |X|^j [N_{1,0}, \Omega] \rangle \\ &\quad - \mu \{ \langle |X|^{j+3} \rangle d_\Omega^2 (M - L) + 3 \langle |X|^{j+1} \rangle d_\Omega (M - L) \} \\ &\quad + \hat{v}_{\varphi i} \{ \langle |X|^{j+1} \rangle (M - L) + v \langle |X|^j \rangle \}, \end{aligned} \quad (68)$$

outside the separatrix, and

$$\langle |X|^j G \rangle = \hat{v}_{\varphi i} v \langle |X|^j \rangle, \quad (69)$$

inside the separatrix.

Equation (66) can be integrated in Ω to give

$$\begin{aligned} 0 &= \frac{d}{d\Omega} \left[\langle X^4 \rangle d_\Omega (M - L) + \frac{1}{2} \frac{D}{\mu} \langle \widetilde{X^2 X^2} \rangle d_\Omega L \right] - \frac{1}{2} \frac{D}{\mu} \langle \widetilde{X^2 X^2} \rangle \\ &\quad \times \{ (1 + 2\tau)d_\Omega M - \tau d_\Omega L \} \frac{d_\Omega L}{M + \tau L} - (1 + \tau)(\alpha_c - \alpha_n) \\ &\quad \times \frac{D}{\mu} \langle \widetilde{|X|X^2} \rangle \frac{d_\Omega L}{M + \tau L} - \frac{\hat{v}_{\varphi i}}{\mu} \{ \langle X^2 \rangle (M - L) + v \}, \end{aligned} \quad (70)$$

outside the separatrix, where use has been made of Eqs. (42), (64), (65), and (68).

Finally, Eq. (67) can be integrated in Ω to give

$$\begin{aligned} &\int_{-1}^{\infty} \langle X[J_{1,0}, \Omega] \rangle d\Omega \\ &= \int_{-1}^{\infty} \langle G \rangle d\Omega = \hat{v}_{\varphi i} \int_{-1}^{\infty} (M - L + v \langle 1 \rangle) d\Omega, \end{aligned} \quad (71)$$

where use has been made of Eqs. (42), (64), (65), (68), and (69).

D. Separatrix boundary layer

The flux-surface functions $M(\Omega)$ and $L(\Omega)$ are both zero inside, and nonzero just outside, the magnetic separatrix. The apparent discontinuities in these two functions are resolved in a thin boundary layer on the separatrix of (un-normalized) width ρ_i .⁴⁹ Inside this layer, Eq. (70) reduces to

$$\begin{aligned} 0 &\approx \frac{d^2}{dy^2} \left(M - L + \frac{1}{2} \frac{D}{\mu} L \right) \\ &\quad - \frac{1}{2} \frac{D}{\mu} d_y \{ (1 + 2\tau)M - \tau L \} \frac{d_y L}{M + \tau L}, \end{aligned} \quad (72)$$

where $y = (\Omega - 1)/(\rho_i/w)$, and $d_y \equiv d/dy$. Note that $d_y \sim \mathcal{O}(w/\rho_i) \gg 1$. Here, we have made use of the fact that $\langle X^4 \rangle = \langle \widetilde{X^2 X^2} \rangle$ close to the separatrix. Let us assume that $M = (1 - v_0)L$ within the layer, where v_0 is a constant. It follows that

$$\begin{aligned} 0 &= \frac{d}{dy} \left(L \frac{dL}{dy} \right) - \left[\frac{v_0^2 - (1 + \tau)(1 + D/\mu)v_0 + (1 + \tau)D/\mu}{(v_0 - 1 - \tau)(v_0 - D/2\mu)} \right] \\ &\quad \times \left(\frac{dL}{dy} \right)^2. \end{aligned} \quad (73)$$

Integrating across the layer from just inside the separatrix (i.e., $y \rightarrow -\infty$, where $L=0$) to just outside the separatrix [i.e., $y \rightarrow \infty$, where $d_y L \sim \mathcal{O}(\rho_i/w) \ll 1$, since $d_\Omega L \sim \mathcal{O}(1)$], we obtain

$$\begin{aligned} &\left| \frac{v_0^2 - (1 + \tau)(1 + D/\mu)v_0 + (1 + \tau)D/\mu}{(v_0 - 1 - \tau)(v_0 - D/2\mu)} \right| \\ &\quad \times \int_{-\infty}^{\infty} \left(\frac{dL}{dy} \right)^2 dy \ll 1. \end{aligned} \quad (74)$$

Now, the integral in the above expression is positive definite, and also much larger than unity. Thus, the only way in which Eq. (74) can be satisfied is if

$$v_0^2 - (1 + \tau)(1 + D/\mu)v_0 + (1 + \tau)D/\mu \approx 0, \quad (75)$$

which implies that

$$v_0 = \frac{(1 + \tau)}{2} \left(1 + \frac{D}{\mu} - \left[1 - 2 \frac{D}{\mu} \left(\frac{1 - \tau}{1 + \tau} \right) + \left(\frac{D}{\mu} \right)^2 \right]^{1/2} \right). \quad (76)$$

Here, we have chosen the root of the quadratic equation (75) which corresponds to the obvious physical solution $v_0 = 0$ when $D/\mu = 0$.⁵⁸ Note that $0 \leq v_0 \leq 1$.

E. Determination of flow profiles

It is convenient to define a new flux-surface label $k \equiv [(1 + \Omega)/2]^{1/2}$. Thus, $k=0$ corresponds to the O-points of the island chain, $k=1$ to the X-points and the magnetic separatrix, and $k \rightarrow \infty$ to $|X| \rightarrow \infty$. It is also helpful to define the complete elliptic integrals

$$E(k) \equiv \int_0^{\pi/2} (1 - k^2 \sin^2 u)^{1/2} du, \quad (77)$$

$$K(k) \equiv \int_0^{\pi/2} (1 - k^2 \sin^2 u)^{-1/2} du. \tag{78}$$

It is easily demonstrated that

$$\langle 1 \rangle = \begin{cases} K(k)/\pi & 0 \leq k < 1 \\ K(1/k)/(k\pi) & 1 < k, \end{cases} \tag{79}$$

$$\langle X^2 \rangle = \begin{cases} (4/\pi)E(k) + (4/\pi)(k^2 - 1)K(k) & 0 \leq k < 1 \\ (4k/\pi)E(1/k) & 1 \leq k, \end{cases} \tag{80}$$

$$\langle |X^3| \rangle = 2(2k^2 - 1) \quad 1 \leq k, \tag{81}$$

$$\langle X^4 \rangle = (16k^4/3\pi)[2(2k^2 - 1)E(1/k) - (k^2 - 1)K(1/k)] \quad 1 < k. \tag{82}$$

Equation (70) reduces to

$$0 = \frac{d}{dk} \left[\frac{\langle X^4 \rangle}{4k} d_k M + \left(1 - \frac{1}{2} \frac{D}{\mu} \right) \frac{\langle X^4 \rangle \langle 1 \rangle}{\langle X^2 \rangle^2} \right] + \frac{1}{2} \frac{D}{\mu} \left(\frac{\langle X^4 \rangle \langle 1 \rangle}{\langle X^2 \rangle^2} - 1 \right) \frac{[(1 + 2\tau)\langle X^2 \rangle d_k M + \tau 4k \langle 1 \rangle / \langle X^2 \rangle]}{\langle X^2 \rangle M + \tau} + (1 + \tau)(\alpha_c - \alpha_n) \frac{D}{\mu} \left(\frac{\langle |X^3| \rangle \langle 1 \rangle}{\langle X^2 \rangle} - 1 \right) \frac{4k}{\langle X^2 \rangle M + \tau}$$

$$- \frac{\hat{v}_{\phi i}}{\mu} 4k[\langle X^2 \rangle M + v - 1], \tag{83}$$

where $d_k \equiv d/dk$. The above equation describes a competition between cross flux-surface momentum transport due to perpendicular ion viscosity (first term on the right-hand side), ion toroidal flow damping (last term on the right-hand side), and coupling to drift waves (all terms proportional to D/μ). (Incidentally, the D/μ terms are associated with coupling to drift waves because they contain the resonant denominator $\langle X^2 \rangle M + \tau \omega - k_\theta (V_E + V_{*e})$, where V_E and V_{*e} are the local equilibrium $\mathbf{E} \times \mathbf{B}$ and electron diamagnetic velocities, respectively.) Equation (83) must be solved for $M(k)$ in the region $1 < k < \infty$, subject to the boundary conditions [see Eq. (42) and Sec. III D]

$$M(k \rightarrow 1) \rightarrow (1 - v_0) \frac{\pi}{4}, \tag{84}$$

$$M(k \rightarrow \infty) \rightarrow \frac{1 - v}{2k}, \tag{85}$$

where v_0 is given by Eq. (76). This procedure fully specifies $M(k)$.

It follows from Eqs. (18), (19), (38)–(41), (44), and (64) that, to lowest order,

$$\frac{V_{\theta i} - V_{\theta i}^{mc}}{V_{*i}} = 0, \tag{86}$$

$$\frac{V_{\phi i} - V_{\phi i}^{mc}}{(q_s / \epsilon_s) V_{*i}} = \begin{cases} -v & 0 \leq k \leq 1 \\ -2[k^2 - \cos^2(\zeta/2)]^{1/2} (M - 1/\langle X^2 \rangle) - v & 1 < k. \end{cases} \tag{87}$$

Thus, in the strong poloidal flow damping regime there is *complete damping* of the ion *poloidal* flow in the vicinity of the island chain. Moreover, in general, the ion toroidal flow, which is not completely damped, is *discontinuous* across the magnetic separatrix (since v_0 is generally nonzero). However, this apparent discontinuity is resolved within the boundary layer described in Sec. III D.

F. Evaluation of cosine integral

The lowest order contribution to the cosine integral, J_c , which is defined in Eq. (21), comes from $J_{0,0}$. Hence,

$$J_c = -2 \int_{-\infty}^{\infty} \oint J_{0,0} \cos \zeta dX \frac{d\zeta}{2\pi} = -4 \int_{-1}^{\infty} \langle J_{0,0} \cos \zeta \rangle d\Omega. \tag{88}$$

It follows from Eqs. (20), (47), (63), and (64) that the island chain's Rutherford equation takes the form

$$\frac{dw}{dt} \propto \Delta' r_s + 2m_\theta \left(\frac{w_v}{w} \right)^2 \cos \phi + I_p \beta_i \left(\frac{q_s}{\epsilon_s} \right)^2 \left(\frac{L_s}{L_n} \right)^2 \times \left(\frac{\rho_i}{r_s} \right)^2 \left(\frac{r_s}{w} \right)^3 + I_b (1 + \tau) \beta_i \left(\frac{q_s}{\epsilon_s} \right) \left(\frac{L_s}{L_n} \right) \times \left[\frac{(\epsilon_s/q_s)^2 v_{\theta e} \tau_e}{1 + (\epsilon_s/q_s)^2 v_{\theta e} \tau_e} - 2 \left(\frac{\epsilon_s}{q_s} \right) \left(\frac{L_s}{L_c} \right) \right] \left(\frac{r_s}{w} \right), \tag{89}$$

where

$$I_p = -\frac{2\pi}{3} v_0 (1 - v_0) + \int_1^\infty \frac{\langle X^2 \rangle}{\langle 1 \rangle} \left(\frac{\langle X^4 \rangle \langle 1 \rangle}{\langle X^2 \rangle^2} - 1 \right) \times [d_k M (\langle X^2 \rangle M - 1) + \langle X^2 \rangle M (d_k M + 4k \langle 1 \rangle / \langle X^2 \rangle^2)] dk, \tag{90}$$

$$I_b = 8 \int_1^\infty \frac{k}{\langle 1 \rangle} \left(\frac{\langle |X^3| \rangle \langle 1 \rangle}{\langle X^2 \rangle} - 1 \right) dk = 1.58. \tag{91}$$

Here, the third term on the right-hand side of Eq. (89) is the

contribution of the *neoclassically enhanced ion polarization current*.³⁷ This contribution is destabilizing if the *polarization integral*, I_p , is positive, and vice versa. [Note that the first term on the right-hand side of the expression (90) for I_p originates from the boundary layer on the magnetic separatrix,⁵⁹ whereas the second term originates from the region immediately outside the separatrix.] The final term on the right-hand side of Eq. (89) is the combined contribution of the perturbed bootstrap current³⁵ and mean magnetic-field line curvature.³⁸ (The bootstrap current contribution [first term in square brackets] is destabilizing, whereas the curvature contribution [second term in square brackets] is stabilizing.)

G. Evaluation of sine integral

The lowest order contribution to the sine integral, J_s , defined in Eq. (23), comes from $J_{1,0}$. Hence,

$$J_s = -2 \int_{-\infty}^{\infty} \oint J_{1,0} \sin \zeta dX \frac{d\zeta}{2\pi} = -4 \int_{-1}^{\infty} \langle X[J_{1,0}, \Omega] \rangle d\Omega. \quad (92)$$

It follows from Eqs. (64) and (71) that

$$J_s = -16 \hat{v}_{\phi i} \left(v \int_0^1 k \langle 1 \rangle dk + \int_1^{\infty} k [M - 1/\langle X^2 \rangle + v \langle 1 \rangle] dk \right). \quad (93)$$

H. Low toroidal flow damping limit

In the *low toroidal flow damping limit*, $\delta \ll \hat{v}_{\phi i} / \mu \ll 1$, the solution to Eqs. (83)–(85) is somewhat simplified. Indeed, far from the separatrix, in the region $1 \ll k$, these equations yield

$$M(k) \approx \frac{1 - v + (v - v_f) e^{-2(\hat{v}_{\phi i} / \mu)^{1/2} k}}{2k}, \quad (94)$$

where v_f is an arbitrary constant. Closer to the separatrix, in the region $1 < k \ll (\hat{v}_{\phi i} / \mu)^{-1/2}$, Eq. (83) reduces to

$$\begin{aligned} 0 &\approx \frac{d}{dk} \left[\frac{\langle X^4 \rangle}{4k} d_k M + \left(1 - \frac{1D}{2\mu} \right) \frac{\langle X^4 \rangle \langle 1 \rangle}{\langle X^2 \rangle^2} \right] + \frac{1D}{2\mu} \\ &\times \left(\frac{\langle X^4 \rangle \langle 1 \rangle}{\langle X^2 \rangle^2} - 1 \right) \frac{[(1 + 2\tau) \langle X^2 \rangle d_k M + \tau 4k \langle 1 \rangle / \langle X^2 \rangle]}{\langle X^2 \rangle M + \tau} \\ &+ (1 + \tau)(\alpha_c - \alpha_n) \frac{D}{\mu} \left(\frac{\langle |X|^3 \rangle \langle 1 \rangle}{\langle X^2 \rangle} - 1 \right) \frac{4k}{\langle X^2 \rangle M + \tau}. \quad (95) \end{aligned}$$

The above equation must be solved in the region $1 < k < \infty$, subject to the boundary conditions

$$M(k \rightarrow 1) \rightarrow (1 - v_0) \frac{\pi}{4}, \quad (96)$$

$$M(k \rightarrow \infty) \rightarrow \frac{1 - v_f}{2k}, \quad (97)$$

where v_0 is given by Eq. (76). This procedure fully specifies

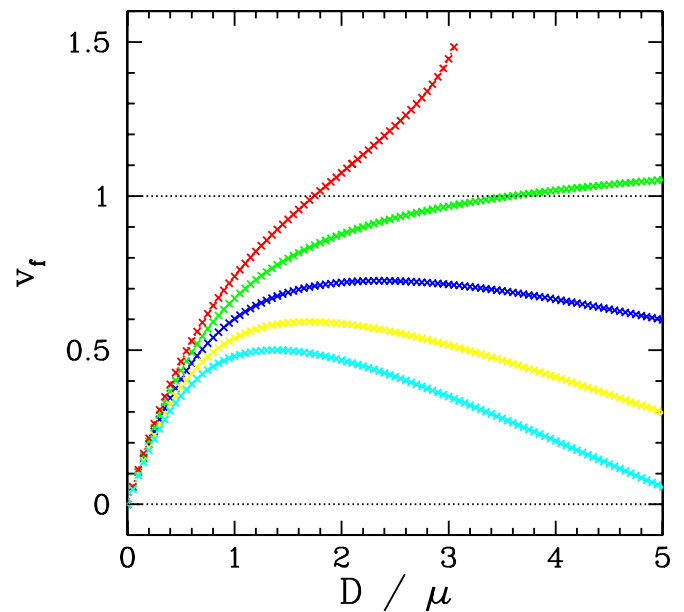


FIG. 1. (Color online) The phase velocity parameter, v_f , for a freely rotating island chain, calculated as a function of the perpendicular diffusivity ratio, D/μ , in the low toroidal flow damping limit of the strong poloidal flow damping regime. The first, second, third, fourth, and fifth curves (in order from the top to the bottom) correspond to $\alpha_c - \alpha_n = -1.0, -0.5, 0.0, 0.5$, and 1.0 , respectively. The calculations are performed with $\tau = 1.0$.

both $M(k)$ and the constant v_f . Now, according to Eqs. (93) and (94),

$$J_s \approx -4(\hat{v}_{\phi i} \mu)^{1/2} (v - v_f) + \mathcal{O}(\hat{v}_{\phi i}). \quad (98)$$

Hence, making use of the criterion (24), we can identify v_f as the characteristic phase velocity parameter for a freely rotating island chain. Now, the polarization integral, I_p , specified in Eq. (90), converges for $k \ll (\hat{v}_{\phi i} / \mu)^{-1/2}$, and can thus be evaluated using the solution to Eq. (95). It follows that I_p depends on the free phase velocity parameter, v_f , but *not* on the actual phase velocity parameter, v . In other words, the contribution of the ion polarization current to the Rutherford equation is the *same* for both a freely rotating and a locked island chain.

Figure 1 shows the phase velocity parameter, v_f , of a freely rotating island chain plotted as a function of D/μ for various different values of $\alpha_c - \alpha_n$. It can be seen that $v_f = 0$ when $D/\mu = 0$, but that $v_f > 0$ when $D/\mu > 0$. This suggests that coupling to drift waves (parameterized by D/μ) causes a freely rotating island chain to propagate in the electron diamagnetic direction, relative to the local unperturbed ion fluid. (The actual phase velocity of the chain is $V_p^{nc} + v_f V_{*i}$, where V_p^{nc} is the neoclassical phase velocity.) Note that the relative propagation velocity decreases with increasing $\alpha_c - \alpha_n$, i.e., the relative velocity is a *decreasing* function of mean magnetic curvature. Figure 2 shows the polarization integral, I_p , plotted as a function of D/μ for various different values of $\alpha_n - \alpha_c$. It can be seen that $I_p = 0$ when $D/\mu = 0$, but that $0 > I_p \gtrsim -1$ when $D/\mu > 0$. This implies that coupling to drift waves allows the neoclassically enhanced ion polarization current to have a relatively *strong stabilizing* effect on both a freely rotating and a locked island chain.

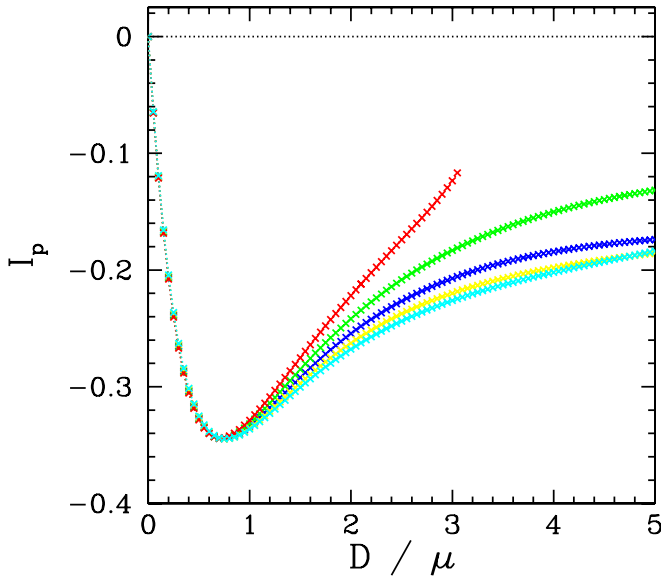


FIG. 2. (Color online) The polarization integral, I_p , calculated as a function of the perpendicular diffusivity ratio, D/μ , in the low toroidal flow damping limit of the strong poloidal flow damping regime. The first, second, third, fourth, and fifth curves (in order from the top to the bottom) correspond to $\alpha_c - \alpha_n = -1.0, -0.5, 0.0, 0.5$, and 1.0 , respectively. The calculations are performed with $\tau = 1.0$.

I. High toroidal flow damping limit

In the *high toroidal flow damping limit*, $1 \ll \hat{v}_{\phi i} / \mu \ll \delta^{-1}$, the solution to Eqs. (83)–(85) is very considerably simplified. In fact, it is easily demonstrated that

$$M(k) \approx \frac{1-v}{\langle X^2 \rangle} + \frac{\pi}{4}(v-v_0)e^{-(6\hat{v}_{\phi i}/\mu)^{1/2}(k-1)}. \quad (99)$$

Hence, Eq. (93) yields

$$\begin{aligned} J_s &\approx -16\hat{v}_{\phi i}v \left[\int_0^1 k\langle 1 \rangle dk + \int_1^\infty k \left(\langle 1 \rangle - \frac{1}{\langle X^2 \rangle} \right) dk \right] \\ &\quad - \sqrt{\frac{8}{3}}\pi(\hat{v}_{\phi i}\mu)^{1/2}(v-v_0) \\ &\approx -5.51\hat{v}_{\phi i}v - 5.13(\hat{v}_{\phi i}\mu)^{1/2}(v-v_0). \end{aligned} \quad (100)$$

Thus, from Eq. (24), the characteristic phase velocity of a freely rotating island chain is

$$v_f \approx 0.93 \left(\frac{\mu}{\hat{v}_{\phi i}} \right)^{1/2} v_0, \quad (101)$$

where v_0 is given by Eq. (76). Note that $v_0 = 0$ when $D/\mu = 0$, but that $0 < v_0 < 1$ when $D/\mu > 0$. It follows that $v_f = 0$ when $D/\mu = 0$, but that $0 < v_f \ll 1$ when $D/\mu > 0$. We conclude that coupling to drift waves causes a freely rotating island to propagate *weakly* in the electron diamagnetic direction, relative to the local unperturbed ion fluid. According to Eq. (90), the polarization integral takes the form

$$\begin{aligned} I_p &\approx v(v-1) \left[\frac{2\pi}{3} - 8 \int_1^\infty \frac{k}{\langle X^2 \rangle} \left(\frac{\langle X^4 \rangle \langle 1 \rangle}{\langle X^2 \rangle^2} - 1 \right) dk \right] \\ &= 1.38v(v-1). \end{aligned} \quad (102)$$

So, for a freely rotating island chain, with $v = v_f$, this expression reduces to

$$I_p \approx -1.28 \left(\frac{\mu}{\hat{v}_{\phi i}} \right)^{1/2} v_0. \quad (103)$$

Note that $I_p = 0$ when $v_0 = 0$, but that $0 > I_p \gg -1$ when $v_0 > 0$. We conclude that coupling to drift waves allows the neoclassically enhanced ion polarization current to have a relatively *weak stabilizing* effect on a freely rotating island chain. Now, for a locked island chain, with $v = v_l$, where v_l is given in Eq. (25), Eq. (102) implies that the polarization current has a stabilizing effect when $0 < v_l < 1$, and a destabilizing effect otherwise. It follows that the polarization current is *stabilizing* provided the locked chain's neoclassical phase velocity, V_p^{nc} , is in the *ion* diamagnetic direction, but is smaller in magnitude than the local ion diamagnetic velocity, i.e., $0 > V_p^{nc} > -V_{*i}$.²²

IV. INTERMEDIATE POLOIDAL FLOW DAMPING REGIME

A. Ordering scheme

The ordering scheme adopted in the so-called *intermediate poloidal flow damping regime* is

$$\Delta^0 \delta^0: v, \tau, \alpha_n,$$

$$\Delta^0 \delta^1: \epsilon, \rho, \beta, \alpha_c,$$

$$\Delta^1 \delta^0: \eta, D, \mu, \hat{v}_{\theta i}, \hat{v}_{\phi i},$$

$$\Delta^1 \delta^2: \hat{v}_{\theta e}.$$

This scheme implies that

$$\omega_{*i} \gg \nu_{\theta i} \gg \nu_{\eta}, \nu_D, \nu_{\mu}, (\epsilon_s/q_s)^2 \nu_{\theta i}, \nu_{\phi i} \gg (m_e/m_i) \nu_{\theta e}. \quad (104)$$

According to the above inequality, the ion poloidal flow damping term is *negligible* to zeroth order in the primary expansion, as are the other flow damping and transport terms. On the other hand, the poloidal flow damping term is dominant in the parallel plasma vorticity equation, (8), to first order in the primary expansion. As before, the above ordering scheme leads to Eqs. (27)–(30). However, Eq. (31) is replaced by

$$L_c \sim \left(\frac{q_s}{\epsilon_s} \right) L_s. \quad (105)$$

B. Zeroth order solution

To lowest order in the primary and secondary expansions (i.e., to order $\Delta^0 \delta^0$), Eqs. (6)–(10) yield

$$0 = [\phi_{0,0} + \tau N_{0,0}, \psi_{0,0}], \quad (106)$$

$$0 = [\phi_{0,0}, N_{0,0}], \quad (107)$$

$$0 = [J_{0,0}, \psi_{0,0}], \quad (108)$$

$$0 = [\phi_{0,0}, V_{0,0}] - (1 + \tau)\alpha_n[N_{0,0}, \psi_{0,0}], \quad (109)$$

$$0 = \partial_X^2 \psi_{0,0} - 1, \quad (110)$$

respectively.

Equations (13) and (110) again lead to Eq. (37). Likewise, Eqs. (14), (15), (106), and (107) reduce to Eqs. (38)–(43). On the other hand, Eqs. (16), (38), (39), and (109) give

$$V_{0,0} = \bar{V}(\Omega), \quad (111)$$

where

$$\bar{V}(\Omega \rightarrow \infty) \rightarrow 0. \quad (112)$$

Finally, Eq. (108) implies that $\bar{J}_{0,0} = 0$. Thus, assuming that $\langle J_{0,0} \rangle = 0$, which turns out to be the case, we obtain

$$J_{0,0} = 0. \quad (113)$$

To zeroth order in the primary expansion, and first order in the secondary expansion (i.e., to order $\Delta^0 \delta^1$), Eq. (8) gives

$$[J_{0,1}, \psi_{0,0}] = -\epsilon \partial_X [\phi_{0,0} - N_{0,0}, \partial_X \phi_{0,0}] - (1 + \tau)\alpha_c [N_{0,0}, X], \quad (114)$$

which implies that

$$J_{0,1} = (\epsilon/2)d_\Omega [M(M-L)]\bar{X}^2 - (1 + \tau)\alpha_c L|\bar{X}| + \bar{J}(\Omega). \quad (115)$$

C. First order solution

To first order in the primary expansion, and zeroth order in the secondary expansion (i.e., to order $\Delta^1 \delta^0$), Eqs. (6)–(10) yield

$$0 = [\phi_{1,0} + \tau N_{1,0}, \psi_{0,0}] + [\phi_{0,0} + \tau N_{0,0}, \psi_{1,0}], \quad (116)$$

$$0 = [\phi_{1,0}, N_{0,0}] + [\phi_{0,0}, N_{1,0}] + D\partial_X^2 N_{0,0}, \quad (117)$$

$$0 = [J_{1,0}, \psi_{0,0}] + \hat{v}_{\theta i} \partial_X \{V_{0,0} - \partial_X(\phi_{0,0} - N_{0,0}) + v\}, \quad (118)$$

$$0 = [\phi_{1,0}, V_{0,0}] + [\phi_{0,0}, V_{1,0}] - (1 + \tau)\alpha_n [N_{1,0}, \psi_{0,0}] - (1 + \tau)\alpha_n [N_{0,0}, \psi_{1,0}] + \mu \partial_X^2 V_{0,0} - \hat{v}_{\theta i} \{V_{0,0} - \partial_X(\phi_{0,0} - N_{0,0}) + v\} - \hat{v}_{\phi i} V_{0,0}, \quad (119)$$

$$0 = \partial_X^2 \psi_{1,0}, \quad (120)$$

respectively. Moreover, Eqs. (120) and (116)–(119) give

$$\psi_{1,0} = 0, \quad (121)$$

$$\phi_{1,0} = -\tau N_{1,0}, \quad (122)$$

$$[N_{1,0}, \Omega] = \frac{D(X^2 d_\Omega L + L)}{M + \tau L}, \quad (123)$$

$$[J_{1,0}, \Omega] = -\hat{v}_{\theta i} \partial_X \{\bar{V} + |X|(M-L) + v\}, \quad (124)$$

$$\begin{aligned} \tau [N_{1,0}, \bar{V}] - sM[V_{1,0}, \Omega] + (1 + \tau)\alpha_n [N_{1,0}, \Omega] \\ = -\hat{v}_{\theta i} \{\bar{V} + |X|(M-L) + v\} + \mu \partial_X^2 \bar{V} - \hat{v}_{\phi i} \bar{V}, \end{aligned} \quad (125)$$

respectively. In addition, the lowest order flux-surface average of Eq. (6) reduces to

$$\beta \eta \bar{J} \langle 1 \rangle = -(1 + \tau)\alpha_n^{-1} \hat{v}_{\theta e} (L - \langle 1 \rangle). \quad (126)$$

Equation (126) implies that

$$\bar{J}(\Omega) = (1 + \tau)\alpha_n \epsilon \nu_{\theta e} \tau_c (1 - L/\langle 1 \rangle). \quad (127)$$

As before, the flux-surface average of Eq. (123), combined with the boundary condition (43), yields expression (64) outside the magnetic separatrix. Equation (125) reduces to

$$\bar{V}(\Omega) = -\left(\frac{\hat{v}_{\theta i}}{\hat{v}_{\theta i} + \hat{v}_{\phi i}}\right)v, \quad (128)$$

inside the separatrix, assuming that $N_{1,0} = V_{1,0} = 0$ in this region. The flux-surface average of Eq. (124), combined with the boundary condition (112), gives

$$\bar{V}(\Omega) = -\langle X^2 \rangle F(\Omega) - v, \quad (129)$$

outside the separatrix, where $F(\Omega) \equiv M(\Omega) - L(\Omega)$. Assuming that $\bar{V}(\Omega)$ is continuous across the separatrix [because of the perpendicular diffusion operator acting on \bar{V} in Eq. (125) (Ref. 22)], we obtain

$$F(\Omega \rightarrow 1) \rightarrow -\frac{\pi}{4} \left(\frac{\hat{v}_{\phi i}}{\hat{v}_{\theta i} + \hat{v}_{\phi i}}\right)v. \quad (130)$$

Likewise, the boundary conditions (42) and (43) give

$$F(\Omega \rightarrow \infty) \rightarrow -\frac{v}{\sqrt{2\Omega}}. \quad (131)$$

Of course, $F=0$ inside the separatrix. The discontinuity in $F(\Omega)$ across the separatrix is resolved in a thin boundary layer of the type discussed in Sec. III D. Now, the flux-surface average of Eq. (125) yields

$$0 = \frac{d}{d\Omega} [\langle X^2 \rangle d_\Omega (\langle X^2 \rangle F)] - \frac{\hat{v}_{\theta i}}{\mu} (\langle X^2 \rangle \langle 1 \rangle - 1) F - \frac{\hat{v}_{\phi i}}{\mu} \langle 1 \rangle \times (\langle X^2 \rangle F + v), \quad (132)$$

outside the separatrix. Moreover, the flux-surface average of X times Eq. (124) gives

$$\langle X [J_{1,0}, \Omega] \rangle = -d_\Omega \langle X^2 G \rangle + \langle G \rangle, \quad (133)$$

where

$$G(\Omega, \zeta) = \begin{cases} \hat{v}_{\theta i} [\hat{v}_{\phi i} / (\hat{v}_{\theta i} + \hat{v}_{\phi i})] v & -1 \leq \Omega \leq 1 \\ -\hat{v}_{\theta i} (\langle X^2 \rangle - |X|) F(\Omega) & 1 < \Omega. \end{cases} \quad (134)$$

Finally, Eq. (133) can be integrated in Ω to give

$$\begin{aligned} \int_{-1}^{\infty} \langle X[J_{1,0}, \Omega] \rangle d\Omega &= \int_{-1}^{\infty} \langle G \rangle d\Omega \\ &= \hat{v}_{\theta i} \left[\left(\frac{\hat{v}_{\phi i}}{\hat{v}_{\theta i} + \hat{v}_{\phi i}} \right) v \int_{-1}^1 \langle 1 \rangle d\Omega \right. \\ &\quad \left. - \int_1^{\infty} (\langle X^2 \rangle \langle 1 \rangle - 1) F d\Omega \right], \end{aligned} \quad (135)$$

where use has been made of (131).

D. Determination of flow profiles

Making use of some of the definitions in Sec. III E, Eq. (132) reduces to

$$\begin{aligned} 0 &= \frac{d}{dk} \left(A(k) \frac{d[A(k)Y(k)]}{dk} \right) - \frac{4\hat{v}_{\theta i}}{\mu} [A(k)B(k) - 1]Y(k) \\ &\quad - \frac{4\hat{v}_{\phi i}}{\mu} B(k)[A(k)Y(k) + v], \end{aligned} \quad (136)$$

where $Y(k) \equiv 2kF(k)$, $A(k) \equiv \langle X^2 \rangle / (2k) = (2/\pi)E(1/k)$, and $B(k) \equiv 2k\langle 1 \rangle = (2/\pi)K(1/k)$. The above equation describes a competition between cross flux-surface momentum transport due to perpendicular ion viscosity (first term on the right-hand side), ion poloidal flow damping (second term on the right-hand side), and ion toroidal flow damping (third term on the right-hand side). Equation (136) must be solved for $Y(k)$ in the region $1 < k < \infty$, subject to the boundary conditions [see Eqs. (130) and (131)]

$$Y(k \rightarrow 1) \rightarrow -\frac{\pi}{2} \left(\frac{\hat{v}_{\phi i}}{\hat{v}_{\theta i} + \hat{v}_{\phi i}} \right) v, \quad (137)$$

$$Y(k \rightarrow \infty) \rightarrow -v. \quad (138)$$

This procedure fully specifies $Y(k)$ [and, hence, $F(k)$].

It follows from Eqs. (18), (19), (38)–(41), (64), (128), and (129) that

$$\frac{V_{\theta i} - V_{\theta i}^{nc}}{V_{*i}} = \begin{cases} [\hat{v}_{\phi i} / (\hat{v}_{\theta i} + \hat{v}_{\phi i})] v & 1 \leq k \leq 1 \\ ([1 - \cos^2(\zeta/2)/k^2]^{1/2} - A(k))Y(k) & 1 < k, \end{cases} \quad (139)$$

$$\frac{V_{\phi i} - V_{\phi i}^{nc}}{(q_s / \epsilon_s) V_{*i}} = \begin{cases} -[\hat{v}_{\theta i} / (\hat{v}_{\theta i} + \hat{v}_{\phi i})] v & 1 \leq k \leq 1 \\ -(A(k)Y(k) + v) & 1 < k. \end{cases} \quad (140)$$

Thus, in the intermediate poloidal flow damping regime there is incomplete damping of both the ion poloidal flow and the ion toroidal flow in the vicinity of the island chain. Moreover, in general, the ion poloidal flow is discontinuous across the separatrix, whereas the ion toroidal flow is continuous. Of course, the apparent discontinuity in $V_{\theta i}$ is resolved within a boundary layer of the type described in Sec. III D.

E. Evaluation of cosine integral

The lowest order contribution to the cosine integral, J_c , comes from $J_{0,1}$. It follows from Eqs. (20), (64), (88), (115), and (127) that the island chain's Rutherford equation takes the form

$$\begin{aligned} \frac{dw}{dt} &\propto \Delta' r_s + 2m_\theta \left(\frac{w_v}{w} \right)^2 \cos \phi + I_p \beta_i \left(\frac{L_s}{L_n} \right)^2 \left(\frac{\rho_i}{r_s} \right)^2 \left(\frac{r_s}{w} \right)^3 \\ &\quad + I_b (1 + \tau) \beta_i \left(\frac{\epsilon_s}{q_s} \right) \left(\frac{L_s}{L_n} \right) \left[(v_{\theta e} \tau_e) - 2 \left(\frac{q_s}{\epsilon_s} \right) \left(\frac{L_s}{L_c} \right) \right] \\ &\quad \times \left(\frac{r_s}{w} \right), \end{aligned} \quad (141)$$

where

$$\begin{aligned} I_p &= \frac{32}{3\pi} [F(F + \pi/4)]_{k=1} + \int_1^{\infty} \frac{\langle X^2 \rangle}{\langle 1 \rangle} \left(\frac{\langle X^4 \rangle \langle 1 \rangle}{\langle X^2 \rangle^2} - 1 \right) \\ &\quad \times [d_k F(\langle X^2 \rangle F + 1) + \langle X^2 \rangle F(d_k F - 4k\langle 1 \rangle / \langle X^2 \rangle^2)] dk, \end{aligned} \quad (142)$$

and $I_b = 1.58$. As before, the first term on the right-hand side of the above expression comes from the boundary layer on the magnetic separatrix, whereas the second term comes from the region immediately outside the separatrix. It can be seen, by comparison with Eqs. (89)–(91), that in the intermediate poloidal flow damping regime the ion polarization term appearing in the Rutherford equation is *smaller* in magnitude by a factor of order $(\epsilon_s/q_s)^2$ than the corresponding term obtained in the strong poloidal flow damping regime. This is due to the absence of any neoclassical enhancement of ion inertia in the intermediate damping regime. On the other hand, the perturbed bootstrap current and mean magnetic curvature terms appearing in the Rutherford equation are essentially the same in both regimes.

F. Evaluation of sine integral

The lowest order contribution to the sine integral, J_s , comes from $J_{1,0}$. It follows from Eqs. (64), (92), and (135) that

$$\begin{aligned} J_s &= -8\hat{v}_{\theta i} \left[\left(\frac{\hat{v}_{\phi i}}{\hat{v}_{\theta i} + \hat{v}_{\phi i}} \right) v \int_0^1 C(k) dk \right. \\ &\quad \left. - \int_1^{\infty} [A(k)B(k) - 1]Y(k) dk \right], \end{aligned} \quad (143)$$

where $C(k) \equiv 2k\langle 1 \rangle = (2/\pi)kK(k)$.

G. Low toroidal flow damping limit

In the low toroidal flow damping limit, $\delta \ll \hat{v}_{\phi i} \ll \mu$, $\mu^2 / \hat{v}_{\theta i} \ll \delta^{-1}$, Eqs. (136)–(138) yield⁵⁵

$$Y(k) \approx -\frac{1}{A(k)} \left[1 - \left(\frac{\hat{v}_{\theta i}}{\hat{v}_{\theta i} + \hat{v}_{\phi i}} \right) e^{-(k-1)/\delta_1} \right] v, \quad (144)$$

where $\delta_1 = (\mu/4\hat{v}_{\phi i})^{1/2} \gg 1$. It follows from Eq. (143) that⁵⁵

$$\begin{aligned} J_s &\approx -8\hat{v}_{\theta i} u \left\{ \int_0^1 C(k) dk + \int_1^{\infty} \left[\frac{A(k)B(k) - 1}{A(k)} \right] dk \right\} \\ &= -5.51 \hat{v}_{\theta i} u, \end{aligned} \quad (145)$$

where

$$u = \left(\frac{\hat{v}_{\varphi i}}{\hat{v}_{\varphi i} + \hat{v}_{\theta i}} \right) v. \quad (146)$$

Hence, from (24), the phase velocity parameter of a freely rotating island chain is $v_f \approx 0$. In other words, a freely rotating island chain does not propagate relative to the local equilibrium ion fluid, i.e., it propagates at the neoclassical phase velocity. According to Eq. (142),⁵⁵

$$\begin{aligned} I_p &\approx u(u-1) \left[\frac{2\pi}{3} - 8 \int_1^\infty \frac{k}{\langle X^2 \rangle} \left(\frac{\langle X^4 \rangle \langle 1 \rangle}{\langle X^2 \rangle^2} - 1 \right) dk \right] \\ &= 1.38u(u-1). \end{aligned} \quad (147)$$

Thus, for a freely rotating island chain, with $v = v_f \approx 0$, the polarization integral is zero. It follows that the ion polarization current has a negligible effect on the stability of a freely rotating island chain. On the other hand, for a locked island chain, with $v = v_l$ [see Eq. (25)], the above expression reduces to

$$I_p \approx 1.38u_l(u_l - 1), \quad (148)$$

where

$$u_l = - \left(\frac{\hat{v}_{\varphi i}}{\hat{v}_{\varphi i} + \hat{v}_{\theta i}} \right) \frac{V_p^{nc}}{V_{*i}}. \quad (149)$$

Note that $I_p < 0$ when $0 < u_l < 1$. We conclude that the ion polarization current has a stabilizing effect on a locked island chain provided the chain's neoclassical phase velocity lies in the range $0 > V_p^{nc} > -(1 + \hat{v}_{\theta i}/\hat{v}_{\varphi i})V_{*i}$, and has a *destabilizing* effect otherwise.

H. Intermediate toroidal flow damping limit

In the *intermediate toroidal flow damping limit*, $\delta \ll \mu^2 / \hat{v}_{\theta i} \ll \hat{v}_{\varphi i} \ll \hat{v}_{\theta i} \ll \delta^{-1}$, Eqs. (136)–(138) yield⁵⁵

$$Y(k) \approx - \frac{\hat{v}_{\varphi i}}{\hat{v}_{\theta i}} \left[\frac{B(k)}{A(k)B(k) - 1 + \hat{v}_{\varphi i}/\hat{v}_{\theta i}} \right] v. \quad (150)$$

It follows from Eq. (143) that⁵⁵

$$J_s \approx -0.617 \hat{v}_{\theta i}^{1/4} \hat{v}_{\varphi i}^{3/4} v. \quad (151)$$

This again implies that $v_f \approx 0$, i.e., a freely rotating island chain propagates at the neoclassical phase velocity. According to Eq. (142),⁵⁵

$$I_p \approx 0.617 \left(\frac{\hat{v}_{\varphi i}}{\hat{v}_{\theta i}} \right)^{3/4} v(v/4 - 1). \quad (152)$$

Hence, we conclude that the ion polarization current has a negligible effect on the stability of a freely rotating island chain. On the other hand, the polarization current has a stabilizing effect on a locked island chain provided the chain's neoclassical phase velocity lies in the range $0 > V_p^{nc} > -4V_{*i}$, and has a destabilizing effect otherwise.

I. High toroidal flow damping limit

In the *high toroidal flow damping limit*, $\delta \ll \mu$, $\hat{v}_{\theta i} \ll \hat{v}_{\varphi i} \ll \delta^{-1}$, Eqs. (136)–(138) yield⁵⁵

$$Y(k) \approx - \frac{1}{A(k)} \left(1 - \frac{\hat{v}_{\theta i}}{\hat{v}_{\varphi i}} \left[1 - \frac{1}{A(k)B(k)} \right] \right) v. \quad (153)$$

It follows from Eq. (143) that⁵⁵

$$J_s \approx -5.51 \hat{v}_{\theta i} v. \quad (154)$$

As before, this implies that $v_f \approx 0$, i.e., a freely rotating island chain propagates at the neoclassical phase velocity. According to Eq. (142),⁵⁵

$$I_p \approx 1.38v(v-1). \quad (155)$$

Hence, we conclude that the ion polarization current has a negligible effect on the stability of a freely rotating island chain. Moreover, the polarization current has a stabilizing effect on a locked island chain provided the chain's neoclassical phase velocity lies in the range $0 > V_p^{nc} > -V_{*i}$, and has a destabilizing effect otherwise.

V. WEAK POLOIDAL FLOW DAMPING REGIME

A. Ordering scheme

The ordering scheme adopted in the so-called *weak poloidal flow damping regime* is

$$\Delta^0 \delta^0: v, \tau, \alpha_n,$$

$$\Delta^0 \delta^1: \epsilon, \rho, \beta, \alpha_c,$$

$$\Delta^1 \delta^0: \eta, D, \mu, \hat{v}_{\varphi i},$$

$$\Delta^1 \delta^1: \hat{v}_{\theta i},$$

$$\Delta^1 \delta^2: \hat{v}_{\theta e}.$$

This scheme implies that

$$\omega_{*i} \gg \nu_\eta, \nu_D, \nu_\mu, \nu_{\theta i}, \nu_{\varphi i} \gg (\epsilon_s/q_s)^2 \nu_{\theta i}, (m_e/m_i) \nu_{\theta e}. \quad (156)$$

According to the above inequality, the ion poloidal flow damping term is negligible to zeroth order in the primary expansion, as are the other flow damping and transport terms. Moreover, the poloidal flow damping term is negligible in the parallel equation of motion (9) to first order in the primary expansion. The above ordering scheme leads to Eqs. (27)–(30) and Eq. (105).

B. Zeroth order solution

To lowest order in the primary and secondary expansions (i.e., to order $\Delta^0 \delta^0$), Eqs. (6)–(10) reduce to exactly the same expressions as those obtained in the intermediate flow damping regime (see Sec. IV B).

C. First order solution

To first order in the primary expansion, and zeroth order in the secondary expansion (i.e., to order $\Delta^1 \delta^0$), Eqs. (6)–(10) yield

$$0 = [\phi_{1,0} + \tau N_{1,0}, \psi_{0,0}] + [\phi_{0,0} + \tau N_{0,0}, \psi_{1,0}], \quad (157)$$

$$0 = [\phi_{1,0}, N_{0,0}] + [\phi_{0,0}, N_{1,0}] + D \partial_X^2 N_{0,0}, \quad (158)$$

$$0 = [J_{1,0}, \psi_{0,0}], \quad (159)$$

$$0 = [\phi_{1,0}, V_{0,0}] + [\phi_{0,0}, V_{1,0}] - (1 + \tau)\alpha_n[N_{1,0}, \psi_{0,0}] - (1 + \tau)\alpha_n[N_{0,0}, \psi_{1,0}] + \mu\partial_X^2 V_{0,0} - \hat{v}_{\phi i} V_{0,0}, \quad (160)$$

$$0 = \partial_X^2 \psi_{1,0}, \quad (161)$$

respectively. Moreover, Eqs. (161) and (157)–(160) give

$$\psi_{1,0} = 0, \quad (162)$$

$$\phi_{1,0} = -\tau N_{1,0}, \quad (163)$$

$$[N_{1,0}, \Omega] = \frac{D(X^2 d_\Omega L + L)}{M + \tau L}, \quad (164)$$

$$[J_{1,0}, \Omega] = 0, \quad (165)$$

$$\begin{aligned} \tau[N_{1,0}, \bar{V}] - sM[V_{1,0}, \Omega] + (1 + \tau)\alpha_n[N_{1,0}, \Omega] \\ = \mu\partial_X^2 \bar{V} - \hat{v}_{\phi i} \bar{V}, \end{aligned} \quad (166)$$

respectively. In addition, the lowest order flux-surface average of Eq. (6) reduces to Eq. (126).

As before, Eq. (126) implies Eq. (127). Moreover, the flux-surface average of Eq. (164), combined with the boundary condition (43), yields expression (64) outside the magnetic separatrix. Equation (164) also gives Eq. (65) outside the separatrix. Now, the flux-surface average of Eq. (166) reduces to

$$\frac{d}{d\Omega} \left(\langle X^2 \rangle \frac{d\bar{V}}{d\Omega} \right) - \frac{\hat{v}_{\phi i}}{\mu} \langle 1 \rangle \bar{V} = 0. \quad (167)$$

The above equation must be solved in the region $-1 < \Omega < \infty$, subject to the boundary conditions that $\bar{V}(\Omega)$ be well-behaved as $\Omega \rightarrow -1$, and that [see Eq. (112)]

$$\bar{V}(\Omega \rightarrow \infty) \rightarrow 0. \quad (168)$$

However, the solution is obvious, i.e.,

$$\bar{V}(\Omega) = 0. \quad (169)$$

Finally, Eq. (165) implies that

$$J_{1,0} = 0. \quad (170)$$

To first order in both the primary and secondary expansions (i.e., to order $\Delta^1 \delta^1$), Eqs. (8), (163), and (169), give

$$[J_{1,1}, \Omega] = -(1 + \tau)\alpha_c[N_{1,0}, X] - \partial_X G, \quad (171)$$

where

$$\begin{aligned} G = (1 + \tau)\epsilon[N_{1,0}, |X|M] - \tau\epsilon(M - L)[|X|\partial_\Omega N_{1,0}, \Omega] \\ - \epsilon\mu\{|X|^3 d_\Omega^2(M - L) + 3|X|d_\Omega(M - L)\} \\ + \hat{v}_{\theta i}\{|X|(M - L) + v\}. \end{aligned} \quad (172)$$

The flux-surface average of Eq. (171) yields

$$0 = d_\Omega \langle |X|G \rangle + (1 + \tau)\alpha_c \langle |X|[N_{1,0}, \Omega] \rangle, \quad (173)$$

outside the separatrix, whereas the flux-surface average of X times this equation gives

$$\begin{aligned} \langle X[J_{1,1}, \Omega] \rangle = -d_\Omega \langle X^2 G \rangle + (1/2)(1 + \tau)\alpha_c \langle X^2 [N_{1,0}, \Omega] \rangle \\ + \langle G \rangle. \end{aligned} \quad (174)$$

Now, it can be demonstrated that

$$\begin{aligned} \langle |X|^j G \rangle = \epsilon(1 + j)^{-1} \{ (1 + \tau)M - j\tau(M - L) \} \\ \times d_\Omega \langle |X|^{j+1} [N_{1,0}, \Omega] \rangle + \epsilon(1 + \tau)d_\Omega M \langle |X|^{j+1} \\ \times [N_{1,0}, \Omega] \rangle + \epsilon j\tau(M - L) \langle |X|^{j-1} [N_{1,0}, \Omega] \rangle \\ - \epsilon\mu \{ \langle |X|^{j+3} \rangle d_\Omega^2(M - L) + 3 \langle |X|^{j+1} \rangle d_\Omega(M - L) \} \\ + \hat{v}_{\theta i} \{ \langle |X|^{j+1} \rangle (M - L) + v \langle |X|^j \rangle \}, \end{aligned} \quad (175)$$

outside the separatrix, and

$$\langle |X|^j G \rangle = \hat{v}_{\theta i} v \langle |X|^j \rangle, \quad (176)$$

inside the separatrix.

Equation (173) can be integrated in Ω to give

$$\begin{aligned} 0 = \frac{d}{d\Omega} \left[\langle X^4 \rangle d_\Omega(M - L) - \frac{1}{2} \frac{D}{\mu} \langle \widetilde{X^2 X^2} \rangle d_\Omega L \right] \\ - \frac{1}{2} \frac{D}{\mu} \langle \widetilde{X^2 X^2} \rangle \{ (1 + 2\tau)d_\Omega M - \tau d_\Omega L \} \frac{d_\Omega L}{M + \tau L} \\ - (1 + \tau) \frac{\alpha_c D}{\epsilon \mu} \langle \widetilde{|X| X^2} \rangle \frac{d_\Omega L}{M + \tau L} - \frac{\hat{v}_{\theta i}}{\epsilon \mu} \{ \langle X^2 \rangle (M - L) + v \}, \end{aligned} \quad (177)$$

outside the separatrix, where use has been made of Eqs. (42), (64), (65), and (175).

Finally, Eq. (174) can be integrated in Ω to give

$$\int_{-1}^{\infty} \langle X[J_{1,1}, \Omega] \rangle d\Omega = \int_{-1}^{\infty} \langle G \rangle d\Omega = \hat{v}_{\theta i} \int_{-1}^{\infty} (M - L + v \langle 1 \rangle) d\Omega, \quad (178)$$

where use has been made of Eqs. (42), (64), (65), (175), and (176).

D. Separatrix boundary layer

The flux-surface functions $M(\Omega)$ and $L(\Omega)$ are both zero inside, and nonzero just outside, the magnetic separatrix. As

before, the apparent discontinuities in these two functions are resolved in a thin boundary layer on the separatrix of (un-normalized) width ρ_i . Inside the layer, Eq. (177) reduces to

$$0 \approx \frac{d^2}{dy^2} \left(M - L - \frac{1}{2} \frac{D}{\mu} L \right) - \frac{1}{2} \frac{D}{\mu} d_y \{ (1 + 2\tau)M - \tau L \} \frac{d_y L}{M + \tau L}, \quad (179)$$

where $y = (\Omega - 1)/(\rho_i/w)$, and $d_y \equiv d/dy$. Let us assume that $M = (1 - v_0)L$ within the layer, where v_0 is a constant. It follows that

$$0 = \frac{d}{dy} \left(L \frac{dL}{dy} \right) - \left[\frac{v_0(v_0 - 1 - \tau[1 + D/\mu])}{(v_0 - 1 - \tau)(v_0 + D/2\mu)} \right] \left(\frac{dL}{dy} \right)^2. \quad (180)$$

Integrating across the layer from just inside the separatrix (i.e., $y \rightarrow -\infty$, where $L=0$) to just outside the separatrix [i.e., $y \rightarrow \infty$, where $d_y L \sim \mathcal{O}(\rho_i/w) \ll 1$, since $d_{\Omega} L \sim \mathcal{O}(1)$], we obtain

$$\left| \frac{v_0(v_0 - 1 - \tau[1 + D/\mu])}{(v_0 - 1 - \tau)(v_0 + D/2\mu)} \right| \int_{-\infty}^{\infty} \left(\frac{dL}{dy} \right)^2 dy \ll 1. \quad (181)$$

Now, the integral in the above expression is positive definite, and also much larger than unity. Thus, the only way in which Eq. (181) can be satisfied is if

$$v_0(v_0 - 1 - \tau[1 + D/\mu]) \approx 0, \quad (182)$$

which implies that

$$v_0 = 0. \quad (183)$$

Here, we have chosen the root of the quadratic Eq. (182) which corresponds to the obvious physical solution $v_0=0$ when $D/\mu=0$.⁵⁸

E. Determination of flow profiles

Making use of the definitions in Sec. III E, Eq. (177) reduces to

$$\frac{V_{\theta i} - V_{\theta i}^{nc}}{V_{*i}} = \begin{cases} v & 0 \leq k \leq 1 \\ 2[k^2 - \cos^2(\zeta/2)]^{1/2}(M - 1/\langle X^2 \rangle) + v & 1 < k, \end{cases} \quad (187)$$

$$\frac{V_{\varphi i} - V_{\varphi i}^{nc}}{(q_s/\epsilon_s)V_{*i}} = 0. \quad (188)$$

Thus, in the weak poloidal flow damping regime there is complete damping of the ion *toroidal* flow in the vicinity of the island chain. Moreover, the ion poloidal flow, which is not completely damped, is *continuous* across the magnetic separatrix (since $v_0=0$).

$$0 = \frac{d}{dk} \left[\frac{\langle X^4 \rangle}{4k} d_k M + \left(1 + \frac{1}{2} \frac{D}{\mu} \right) \frac{\langle X^4 \rangle \langle 1 \rangle}{\langle X^2 \rangle^2} \right] + \frac{1}{2} \frac{D}{\mu} \times \left(\frac{\langle X^4 \rangle \langle 1 \rangle}{\langle X^2 \rangle^2} - 1 \right) \frac{[(1 + 2\tau)\langle X^2 \rangle d_k M + \tau 4k \langle 1 \rangle / \langle X^2 \rangle]}{\langle X^2 \rangle M + \tau} + (1 + \tau) \frac{\alpha_c D}{\epsilon \mu} \left(\frac{\langle |X|^3 \rangle \langle 1 \rangle}{\langle X^2 \rangle} - 1 \right) \frac{4k}{\langle X^2 \rangle M + \tau} - \frac{\hat{v}_{\theta i}}{\epsilon \mu} 4k [\langle X^2 \rangle M + v - 1]. \quad (184)$$

The above equation describes a competition between cross flux-surface momentum transport due to perpendicular ion viscosity (first term on the right-hand side), ion poloidal flow damping (last term on the right-hand side), and coupling to drift waves (all terms proportional to D/μ). Note that the origin of the drift-wave coupling in the equations that determine $M(k)$ is *different* in the strong and weak poloidal flow damping regimes. In the former regime, the coupling comes from the $[\phi, V]$ and $[N, \psi]$ terms in the parallel plasma equation of motion (10), as well as the $(1 + \tau)\alpha_c[N, X]$ term in the parallel plasma vorticity equation (9). In the latter regime, the coupling comes from the $\epsilon \partial_X[\phi - N, \partial_X \phi]$ and $(1 + \tau)\alpha_c[N, X]$ terms in the parallel plasma vorticity equation. Equation (184) must be solved for $M(k)$ in the region $1 < k < \infty$, subject to the boundary conditions [see Eq. (42) and Sec. V D]

$$M(k \rightarrow 1) \rightarrow \frac{\pi}{4}, \quad (185)$$

$$M(k \rightarrow \infty) \rightarrow \frac{1 - v}{2k}. \quad (186)$$

This procedure fully specifies $M(k)$.

It follows from Eqs. (18), (19), (38)–(41), (64), and (169) that, to lowest order,

F. Evaluation of cosine integral

The lowest order contribution to the cosine integral, J_c , comes from $J_{0,1}$. It follows from Eqs. (20), (64), (88), (115), (127), and (183) that the Rutherford equation takes the form (141), where

$$I_p = \int_1^\infty \frac{\langle X^2 \rangle}{\langle 1 \rangle} \left(\frac{\langle X^4 \rangle \langle 1 \rangle}{\langle X^2 \rangle^2} - 1 \right) \times [d_k M (\langle X^2 \rangle M - 1) + \langle X^2 \rangle M (d_k M + 4k \langle 1 \rangle / \langle X^2 \rangle^2)] dk, \quad (189)$$

and $I_b = 1.58$. Note the absence of any contribution from the boundary layer on the separatrix to the above expression [cf., Eqs. (90) and (142)]. As before, there is no neoclassical enhancement of ion inertia in the weak poloidal flow damping regime, so the ion polarization term appearing in the Rutherford equation is smaller in magnitude by a factor of order $(\epsilon_s/q_s)^2$ than the corresponding term obtained in the strong poloidal flow damping regime. However, the perturbed bootstrap current and mean magnetic curvature terms appearing in the Rutherford equation are essentially the same in both regimes.

G. Evaluation of sine integral

The lowest order contribution to the sine integral, J_s , comes from $J_{1,1}$. It follows from Eqs. (64), (92), and (178) that

$$J_s = -16 \hat{v}_{\theta i} \left(v \int_0^1 k \langle 1 \rangle dk + \int_1^\infty k [M - 1/\langle X^2 \rangle + v \langle 1 \rangle] dk \right). \quad (190)$$

H. Low poloidal flow damping limit

In the *low poloidal flow damping limit*, $\delta \ll \hat{v}_{\theta i} / \epsilon \mu \ll 1$, the solution to Eqs. (184)–(186) is somewhat simplified. Indeed, far from the separatrix, in the region $1 \ll k$, these equations yield

$$M(k) \approx \frac{1 - v + (v - v_f) e^{-2(\hat{v}_{\theta i} / \epsilon \mu)^{1/2} k}}{2k}, \quad (191)$$

where v_f is an arbitrary constant. Closer to the separatrix, in the region $1 < k \ll (\hat{v}_{\theta i} / \epsilon \mu)^{-1/2}$, Eq. (184) reduces to

$$0 \approx \frac{d}{dk} \left[\frac{\langle X^4 \rangle}{4k} d_k M + \left(1 + \frac{1D}{2\mu} \right) \frac{\langle X^4 \rangle \langle 1 \rangle}{\langle X^2 \rangle^2} \right] + \frac{1D}{2\mu} \times \left(\frac{\langle X^4 \rangle \langle 1 \rangle}{\langle X^2 \rangle^2} - 1 \right) \frac{[(1 + 2\tau) \langle X^2 \rangle d_k M + \tau 4k \langle 1 \rangle / \langle X^2 \rangle]}{\langle X^2 \rangle M + \tau} + (1 + \tau) \frac{\alpha_c D}{\epsilon \mu} \left(\frac{\langle X^3 \rangle \langle 1 \rangle}{\langle X^2 \rangle} - 1 \right) \frac{4k}{\langle X^2 \rangle M + \tau}. \quad (192)$$

The above equation must be solved in the region $1 < k < \infty$, subject to the boundary conditions

$$M(k \rightarrow 1) \rightarrow \frac{\pi}{4}, \quad (193)$$

$$M(k \rightarrow \infty) \rightarrow \frac{1 - v_f}{2k}. \quad (194)$$

This procedure fully specifies both $M(k)$ and the constant v_f . Now, according to Eqs. (190) and (191),

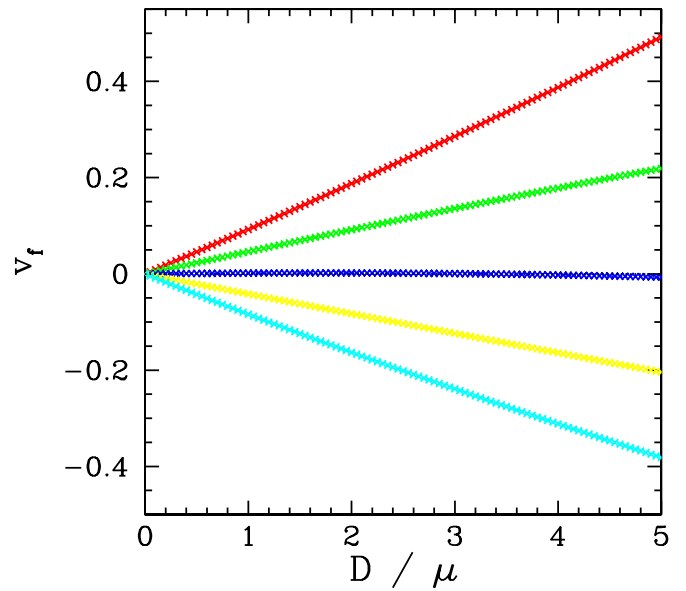


FIG. 3. (Color online) The phase velocity parameter, v_f , for a freely rotating island chain, calculated as a function of the perpendicular diffusivity ratio, D/μ , in the low poloidal flow damping limit of the weak poloidal flow damping regime. The first, second, third, fourth, and fifth curves (in order from the top to the bottom) correspond to $\alpha_c/\epsilon = 0.0, 0.5, 1.0, 1.5$, and 2.0 , respectively. The calculations are performed with $\tau = 1.0$.

$$J_s \approx -4(\hat{v}_{\theta i} \epsilon \mu)^{1/2} (v - v_f) + \mathcal{O}(\hat{v}_{\theta i}). \quad (195)$$

Hence, v_f is the characteristic phase velocity parameter for a freely rotating island chain. Now, the polarization integral, I_p , specified in Eq. (189), converges for $k \ll (\hat{v}_{\theta i} / \epsilon \mu)^{-1/2}$, and can thus be evaluated using the solution to Eq. (192). It follows that I_p depends on the free phase velocity parameter, v_f , but not on the actual phase velocity parameter, v . We thus conclude that the contribution of the ion polarization current to the Rutherford equation is the *same* for both a freely rotating and a locked island chain.

Figure 3 shows the phase velocity parameter, v_f , of a freely rotating island chain plotted as a function of D/μ for various different values of α_c/ϵ . It can be seen that v_f is approximately proportional to D/μ , being positive for $0 < \alpha_c/\epsilon \leq 1$, and negative for $\alpha_c/\epsilon \geq 1$. In other words, coupling to drift waves (parameterized by D/μ) causes a freely rotating island chain to propagate in the electron diamagnetic direction, relative to the local unperturbed ion fluid, when the mean magnetic curvature is relatively *small*, and in the ion diamagnetic direction when the curvature is relatively *large*. Figure 4 shows the polarization integral, I_p , plotted as a function of D/μ for various different values of α_c/ϵ . It can be seen that I_p is also roughly proportional to D/μ . Moreover, I_p is negative when $0 < \alpha_c/\epsilon \leq 1.75$, and positive when $\alpha_c/\epsilon \geq 1.75$. It follows that coupling to drift waves allows the ion polarization current to have a stabilizing effect on either a freely rotating or a locked island chain when the curvature is relatively *small*, and a destabilizing effect when the curvature is relatively *large*.

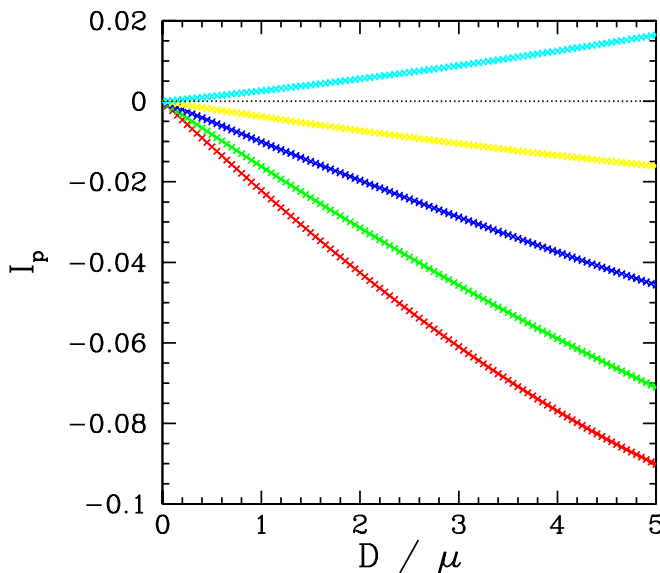


FIG. 4. (Color online) The polarization integral, I_p , calculated as a function of the perpendicular diffusivity ratio, D/μ , in the low poloidal flow damping limit of the weak poloidal flow damping regime. The first, second, third, fourth, and fifth curves (in order from the bottom to the top) correspond to $\alpha_c/\epsilon=0.0, 0.5, 1.0, 1.5,$ and 2.0 , respectively. The calculations are performed with $\tau=1.0$.

I. High poloidal flow damping limit

In the *high toroidal flow damping limit*, $1 \ll \hat{v}_{\theta i}/\epsilon\mu \ll \delta^{-1}$, the solution to Eqs. (184)–(186) is very considerably simplified. In fact, it is easily demonstrated that

$$M(k) \approx \frac{1-v}{\langle X^2 \rangle} + \frac{\pi}{4} v e^{-(6\hat{v}_{\theta i}/\epsilon\mu)^{1/2}(k-1)}. \quad (196)$$

Hence, Eq. (190) yields

$$J_s \approx -16\hat{v}_{\theta i}v \left[\int_0^1 k\langle 1 \rangle dk + \int_1^\infty k \left(\langle 1 \rangle - \frac{1}{\langle X^2 \rangle} \right) dk \right] - \sqrt{\frac{8}{3}} \pi (\hat{v}_{\theta i}/\epsilon\mu)^{1/2} v = -5.51\hat{v}_{\theta i}v - 5.13(\hat{v}_{\theta i}/\epsilon\mu)^{1/2} v. \quad (197)$$

Thus, the characteristic phase velocity of a freely rotating island chain is $v_f \approx 0$, i.e., a freely rotating chain propagates at the neoclassical phase velocity. According to Eq. (189), the polarization integral takes the form

$$I_p \approx v(v-1) \left[\frac{2\pi}{3} - 8 \int_1^\infty \frac{k}{\langle X^2 \rangle} \left(\frac{\langle X^4 \rangle \langle 1 \rangle}{\langle X^2 \rangle^2} - 1 \right) dk \right] = 1.38v(v-1). \quad (198)$$

For a freely rotating island chain, with $v=v_f \approx 0$, this expression reduces to $I_p \approx 0$. It follows that the ion polarization current has a negligible effect on the stability of a freely rotating island chain. On the other hand, for a locked island chain, with $v=v_l$, where v_l is given in Eq. (25), Eq. (198) implies that the ion polarization current has a stabilizing effect when $0 < v_l < 1$, and a destabilizing effect otherwise, i.e., the polarization current is stabilizing provided the

locked chain's neoclassical phase velocity lies in the range $0 > V_p^{nc} > -V_{*i}$.

VI. SUMMARY

We have developed a *drift-MHD fluid model* of an isolated, steady-state, helical magnetic island chain, embedded in the pedestal of a large aspect ratio, low- β , circular cross section, H-mode tokamak plasma, in the presence of an externally generated, multiharmonic, static magnetic perturbation (such as might occur in a typical RMP experiment). The starting point for the model is a set of single helicity, reduced, drift-MHD, fluid equations. These equations can be regarded as an extension of the well-known *four-field equations* of Hazeltine *et al.*⁵⁰ that takes into account neoclassical poloidal and toroidal *flow damping*, as well as the perturbed bootstrap current. The equations are solved analytically in a number of distinct ordering regimes by means of a systematic expansion in small quantities. For the case of a freely rotating island chain, the main aims of the calculation are to determine the chain's *phase velocity*, and the magnitude and sign of the *ion polarization term* in its Rutherford radial width evolution equation. For the case of a locked island chain, the main aims are to determine the magnitude and sign of the polarization term.

The selected ordering regimes are designed to illustrate the effect of various different levels of poloidal flow damping on the island chain, while still ensuring that the constant- ψ approximation holds, and that the lowest order ion flow is tied to perturbed magnetic flux surfaces. The regimes also exploit the peculiar properties of the pedestal of a large aspect ratio, low- β , H-mode tokamak plasma during an RMP experiment, including the relatively *small* density scale length, and the presence of significant toroidal flow damping generated by the nonresonant harmonics of the external magnetic perturbation.

The so-called strong poloidal flow damping regime (see Sec. III) is characterized by

$$\nu_{\theta i} \gg (\epsilon_s/q_s)^2 \nu_{\theta i} \gg \omega_{*i} \gg \nu_\mu, \nu_{\phi i}, \quad (199)$$

where ϵ_s/q_s is the ratio of the poloidal to the toroidal magnetic-field strength, $\nu_{\theta i}$ the ion poloidal flow damping rate, ω_{*i} the ion diamagnetic frequency, ν_μ the rate of radial momentum transport across the island region due to perpendicular ion viscosity, and $\nu_{\phi i}$ the ion toroidal flow damping rate. All quantities are evaluated at the rational surface. According to the above ordering scheme, ion poloidal flow damping is dominant in both the parallel plasma vorticity equation and the parallel plasma equation of motion. This turns out to be necessary condition for obtaining a *neoclassical enhancement of ion inertia*. Ion poloidal flow damping also dominates perpendicular ion viscosity and ion toroidal flow damping in both equations.

The intermediate poloidal flow damping regime (see Sec. IV) is characterized by

$$\omega_{*i} \gg \nu_{\theta i} \gg \nu_\mu, \nu_{\phi i}, (\epsilon_s/q_s)^2 \nu_{\theta i}. \quad (200)$$

According to the above ordering scheme, ion poloidal flow damping is small compared to the leading order terms in

both the parallel plasma vorticity equation and the parallel plasma equation of motion, but dominates perpendicular ion viscosity and ion toroidal flow damping in the former equation, and is the same size as these effects in the latter. There is no neoclassical enhancement of ion inertia in this regime.

Finally, the weak poloidal flow damping regime (see Sec. V) is characterized by

$$\omega_{*i} \gg \nu_{\theta i}, \nu_{\mu}, \nu_{\varphi i} \gg (\epsilon_s/q_s)^2 \nu_{\theta i}. \quad (201)$$

According to the above ordering scheme, ion poloidal flow damping is small compared to the leading order terms in both the parallel plasma vorticity equation and the parallel plasma equation of motion, is the same size as perpendicular ion viscosity and ion toroidal flow damping in the former equation, but is negligible compared to these effects in the latter. Again, there is no neoclassical enhancement of ion inertia in this regime.

In the strong poloidal flow damping regime, the radial electric field profile in the vicinity of the island is determined by a competition between perpendicular ion viscosity, ion toroidal flow damping, and coupling to drift waves (see Sec. III E). Moreover, the ion poloidal velocity in the island region is fully relaxed to its neoclassical value, whereas the ion toroidal velocity is incompletely relaxed. In the intermediate poloidal flow damping regime, the radial electric field profile in the vicinity of the island is determined by a competition between perpendicular ion viscosity, ion poloidal flow damping, and ion toroidal flow damping (see Sec. IV D). Furthermore, neither the ion poloidal velocity nor the ion toroidal velocity in the island region are completely relaxed to their respective neoclassical values. Finally, in the weak poloidal flow damping regime, the radial electric field profile in the vicinity of the island is determined by a competition between perpendicular ion viscosity, ion poloidal flow damping, and coupling to drift waves (see Sec. V E). Moreover, the ion toroidal velocity in the island region is fully relaxed to its neoclassical value, whereas the ion poloidal velocity is incompletely relaxed.

In the strong poloidal flow damping regime, two limits are found, depending on the relative strengths of the ion toroidal flow damping rate, $\nu_{\varphi i}$, and the perpendicular ion momentum diffusion rate, ν_{μ} . The low toroidal flow damping limit corresponds to $\nu_{\varphi i} \ll \nu_{\mu}$ (see Sec. III H). In this limit, coupling to drift waves causes a freely rotating island chain to propagate strongly in the electron diamagnetic direction, relative to the unperturbed local ion fluid. Drift-wave coupling also causes the neoclassically enhanced ion polarization current to have a strong stabilizing effect on both a freely rotating and a locked island chain. The *high toroidal flow damping limit* corresponds to $\nu_{\varphi i} \gg \nu_{\mu}$ (see Sec. IV). In this limit, a freely rotating island chain only propagates weakly in the electron diamagnetic direction, relative to the unperturbed local ion fluid, and the neoclassically enhanced ion polarization current only has a weak stabilizing effect on the chain. For the case of a locked island chain, the ion polarization current is stabilizing when the chain's neoclassical phase velocity (i.e., the fixed value toward which ion poloidal and toroidal flow damping relax the phase velocity) is in the *ion* diamagnetic direction, but is smaller in magni-

tude than the local ion diamagnetic velocity, otherwise it is destabilizing.

In the intermediate poloidal flow damping regime, three limits are found, depending on the relative strengths of the ion poloidal flow damping rate, $\nu_{\theta i}$, the ion toroidal flow damping rate, $\nu_{\varphi i}$, and the perpendicular ion momentum diffusion rate, ν_{μ} . The low toroidal flow damping limit corresponds to $\nu_{\varphi i} \ll \nu_{\mu}$, $\nu_{\mu}^2/(\epsilon\nu_{\theta i})$, where $\epsilon = (\epsilon_s/q_s)^2$ (see Sec. IV G). The intermediate toroidal flow damping limit corresponds to $\nu_{\mu}^2/(\epsilon\nu_{\theta i}) \ll \nu_{\varphi i} \ll \epsilon\nu_{\theta i}$ (see Sec. IV H). Finally, the *strong toroidal flow damping limit* corresponds to ν_{μ} , $\epsilon\nu_{\theta i} \ll \nu_{\varphi i}$ (see Sec. V). In all three regimes, a freely rotating island chain propagates at its neoclassical phase velocity (i.e., it is effectively convected by the unperturbed local ion fluid), and the ion polarization current has a negligible effect on the chain's stability. For the case of a locked island chain, the ion polarization current is stabilizing when the neoclassical phase velocity is in the *ion* diamagnetic direction, but is smaller in magnitude than a critical value which is of order the local ion diamagnetic velocity (but is different in the three different limits), otherwise it is destabilizing.

In the weak poloidal flow damping regime, two limits are found, depending on the relative strengths of the ion poloidal flow damping rate, $\nu_{\theta i}$, and the perpendicular ion momentum diffusion rate, ν_{μ} . The *low poloidal flow damping limit* corresponds to $\nu_{\theta i} \ll \nu_{\mu}$ (see Sec. V H). In this limit, coupling to drift waves causes a freely rotating island chain to propagate in the electron diamagnetic direction, relative to the unperturbed local ion fluid, when the magnetic curvature is relatively small, and in the *ion* diamagnetic direction when the curvature is relatively large. Drift-wave coupling also causes the ion polarization current to have a stabilizing effect on both a freely rotating and a locked island chain when the curvature is relatively small, and a destabilizing effect when the curvature is relatively large. The *high poloidal flow damping limit* corresponds to $\nu_{\theta i} \gg \nu_{\mu}$ (see Sec. VI). In this limit, a freely rotating island chain propagates at its neoclassical phase velocity, and the ion polarization current has a negligible effect on the chain's stability. For the case of a locked island chain, the ion polarization current is stabilizing when the chain's neoclassical phase velocity is in the *ion* diamagnetic direction, but is smaller in magnitude than the local ion diamagnetic velocity, otherwise it is destabilizing.

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