Two-fluid magnetic island dynamics in slab geometry

Richard Fitzpatrick* and François L. Waelbroeck

Institute for Fusion Studies
Department of Physics
University of Texas at Austin
Austin, TX 78712

A novel set of reduced, 2-D, two-fluid, drift-MHD (magnetohydrodynamical) equations is derived. Using these equations, a complete and fully self-consistent solution is obtained for an isolated magnetic island propagating through a slab plasma with uniform but different ion and electron fluid velocities. The ion and electron fluid flow profiles around the island are uniquely determined, and are everywhere continuous. Moreover, the island propagation velocity is uniquely specified by the condition that there be zero net electromagnetic force acting on the island. Finally, the ion polarization current correction to the Rutherford island width evolution equation is evaluated, and found to be stabilizing provided that the anomalous perpendicular ion viscosity significantly exceeds the anomalous perpendicular electron viscosity.

I. INTRODUCTION

Tearing modes are magnetohydrodynamical (MHD) instabilities which often limit fusion plasma performance in magnetic confinement devices relying on nested toroidal magnetic flux-surfaces.¹ As the name suggests, “tearing” modes tear and reconnect magnetic field-lines, in the process converting nested toroidal flux-surfaces into helical magnetic islands.

*rfitzp@farside.ph.utexas.edu
Such islands degrade plasma confinement because heat and particles are able to travel radially from one side of an island to another by flowing along magnetic field-lines, which is a relatively fast process, instead of having to diffuse across magnetic flux-surfaces, which is a relatively slow process.\(^2\)

Magnetic island physics is very well understood within the context of \textit{single-fluid} MHD theory. According to this theory, the island width is governed by the well-known nonlinear evolution equation due to Rutherford.\(^3\) Moreover, the island is also required to propagate at the local flow velocity of the MHD fluid, since fluid flow across the island separatrix is effectively prohibited.

Magnetic island physics is less completely understood within the context of \textit{two-fluid}, drift-MHD theory,\(^4\)–\(^18\) which is far more relevant to present-day magnetic confinement devices than single-fluid theory. In two-fluid theory, the island is generally embedded within ion and electron fluids which flow at \textit{different} velocities. The island itself usually propagates at some intermediate velocity. For sufficiently wide islands, both fluids are required to flow at the island propagation velocity in the region lying within the island separatrix (since neither fluid can easily cross the separatrix). However, the region immediately outside the separatrix is characterized by strongly sheared ion and electron fluid flow profiles, as the velocities of both fluids adjust to their unperturbed values far away from the island. The polarization current generated by the strongly sheared ion flow around the island separatrix generates an additional term in the Rutherford island width evolution equation, which is stabilizing or destabilizing, depending on the island propagation velocity relative to the unperturbed flow velocities of the ion and MHD fluids. The key problems in two-fluid island theory are the unambiguous determination of the island propagation velocity, and the calculation of the ion and electron fluid flow profiles around the island separatrix. As yet, no consensus has emerged within the magnetic fusion community regarding the solution of these problems.

In this paper, we first develop a novel set of reduced, 2-D, two-fluid, drift-MHD equations. These equations contain both electron and ion diamagnetic effects (including the
contribution of the ion gyroviscous tensor), as well as the Hall effect and parallel electron compressibility. However, they do not contain the compressible Alfvén wave (which plays a negligible role in magnetic island physics). Our set of equations consist of four coupled partial differential equations, and is both analytically tractable and easy to solve numerically. We employ our equations to study the evolution of an isolated magnetic island in slab geometry. Using a novel ordering scheme, we are able to calculate the island propagation velocity, and to uniquely determine the ion and electron fluid flow profiles outside the island separatrix.

II. DERIVATION OF REDUCED EQUATIONS

A. Introduction

In this section, we shall generalize the analysis of Refs. 19 and 20 to obtain a set of reduced, 2-D, two-fluid, drift-MHD equations which take ion diamagnetic flows into account.

B. Basic equations

Standard right-handed Cartesian coordinates \((x, y, z)\) are adopted. Consider a quasi-neutral plasma with singly-charged ions. The ion/electron number density \(n_0\) is assumed to be uniform and constant. Suppose that \(T_i = \tau T_e\), where \(T_{i,e}\) is the ion/electron temperature, and \(\tau\) is uniform and constant.

Broadly following Ref. 21, we adopt the following set of two-fluid, drift-MHD equations:

\[
E + V \times B + \frac{1}{e n_0} \left( \nabla P - \frac{\tau}{1 + \tau} (b \cdot \nabla) b - J \times B - \mu_e \nabla^2 V_e \right) = \eta \left( J - \frac{3}{2} \frac{\tau}{1 + \tau} n_0 e V_* \right),
\]

\[
\eta \left[ \left( \frac{\partial}{\partial t} + V \cdot \nabla + \frac{\tau}{1 + \tau} V_e \cdot \nabla \right) V - \frac{\tau}{1 + \tau} V_e \cdot \nabla (b \cdot V) b \right] = J \times B - \nabla P + \mu_i \nabla^2 V_i + \mu_e \nabla^2 V_e,
\]

\[
\left( \frac{\partial}{\partial t} + V \cdot \nabla \right) P = -\Gamma P \nabla \cdot V + \kappa \nabla^2 P.
\]
Here, $E$ is the electric field, $B$ the magnetic field, $J$ the electric current density, $V$ the plasma guiding-center velocity, $P$ the total plasma pressure, $e$ the magnitude of the electron charge, $m_i$ the ion mass, $\eta$ the (uniform) plasma resistivity, $\mu_e$ the (uniform) electron viscosity, $\mu_i$ the (uniform) ion viscosity, $\kappa$ the (uniform) plasma thermal conductivity, and $\Gamma = 5/3$ the plasma ratio of specific heats. Furthermore, $b = B/B$, and $V_e = b \times \nabla P/e n_0 B$. The above equations take into account the anisotropic ion gyroviscous tensor, but neglect electron inertia. Our system of equations is completed by Maxwell’s equations: $\nabla \cdot B = 0$, $\nabla \times E = -\partial B/\partial t$, and $\nabla \times B = \mu_0 J$. Note that the transport coefficients, $\mu_i$, $\mu_e$, and $\kappa$, appearing in the above equations are phenomenological in nature, and are supposed to represent the anomalous diffusive transport of energy and momentum across magnetic flux-surfaces due to small-scale plasma turbulence.

C. Normalized equations

Let $\hat{\nabla} = a \nabla$, $\hat{t} = t/(a/V_a)$, $\hat{B} = B/B_a$, $\hat{E} = E/(B_a V_a)$, $\hat{J} = J/(B_a/\mu_0 a)$, $\hat{V} = V/V_a$, $\hat{P} = P/(B_a^2/\mu_0)$, $\hat{\eta} = \eta/(\mu_0 V_a a)$, $\hat{\mu}_{i,e} = \mu_{i,e}/(n_0 m_i V_a a)$, $\hat{\kappa} = \kappa/(V_a a)$, where $V_a = B_a/\sqrt{\mu_0 n_0 m_i}$. Here, $a$ is a convenient scale length, and $B_a$ a convenient scale magnetic field-strength.

Neglecting hats, our normalized two-fluid equations take the form:

$$E + V \times B + d_i \left( \nabla P - \frac{\tau}{1 + \tau} (b \cdot \nabla P) b - J \times B - \mu_e \nabla^2 V_e \right) = \eta \left( J - \frac{3}{2} \nabla \times V_e \right), \tag{4}$$

$$\left( \frac{\partial}{\partial \hat{t}} + V \cdot \nabla + \frac{d_i \tau}{1 + \tau} V_e \cdot \nabla \right) V = \frac{d_i \tau}{1 + \tau} V_e \cdot \nabla (|b \cdot V| b) = J \times B - \nabla P + \mu_i \nabla^2 V_i + \mu_e \nabla^2 V_e, \tag{5}$$

$$\left( \frac{\partial}{\partial \hat{t}} + V \cdot \nabla \right) P = -\Gamma P \nabla \cdot V + \kappa \nabla^2 P. \tag{6}$$

Here, $V_e = b \times \nabla P/B$, and $d_i = (m_i/n_0 e^2 \mu_0)^{1/2}/a$ is the normalized collisionless ion skin-depth. Maxwell’s equations are written: $\nabla \cdot B = 0$, $\nabla \times E = -\partial B/\partial t$, and $\nabla \times B = J$. 

4
D. 2-D assumption

Let us make the simplifying assumption that there is no variation of quantities in the z-direction: i.e., $\partial / \partial z \equiv 0$. It immediately follows that $B = \nabla \psi \times \hat{z} + B_z \hat{z}$, and $E_z = -\partial \psi / \partial t$.

E. Reduction process

Let us adopt the following ordering, which is designed to decouple the compressional Alfvén wave from all the other waves in the system:

$$P = P_0 + B_0 p_1 + p_2, \quad (7)$$

$$B_z = B_0 + b_z. \quad (8)$$

Here, $P_0$ and $B_0$ are uniform and constant, and

$$P_0 \gg B_0 \gg 1. \quad (9)$$

Furthermore, $p_1$, $p_2$, and $b_z$ are all $O(1)$ quantities. Finally, $\psi$, $V$, $\nabla$, and $\partial / \partial t$ are all assumed to be $O(1)$, whereas $\nabla \cdot V$ is assumed to be much less than $O(1)$.

Now, to lowest order, the $z$-component of Ohm’s law, Eq. (4), gives

$$\left( \frac{\partial}{\partial t} + V \cdot \nabla \right) \psi = -d_i [b_z + \tau p_1/(1 + \tau) \psi] + \eta \nabla^2 \psi - d_i \mu_e \nabla^2 (V_z + d_i \nabla^2 \psi). \quad (10)$$

Here, $[A, B] \equiv \nabla A \times \nabla B \cdot \hat{z}$. Likewise, the $z$-component of the curl of Eq. (4) reduces to

$$\left( \frac{\partial}{\partial t} + V \cdot \nabla \right) b_z = [V_z + d_i \nabla^2 \psi, \psi] - B_0 \nabla \cdot V + \eta \nabla^2 \left( b_z + \frac{3}{2} \frac{\tau}{1 + \tau} p_1 \right)$$

$$+ d_i \mu_e \nabla^2 \left( U - d_i \nabla^2 \left( b_z + \frac{\tau}{1 + \tau} p_1 \right) \right). \quad (11)$$

Here, $U = -\nabla \times V \cdot \hat{z}$.

To lowest order, the equation of motion, Eq. (5), implies that

$$p_1 \simeq -b_z. \quad (12)$$

Furthermore, the $z$-component of this equation yields
\[
\left( \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) V_z = [b_z, \psi] + \mu_i \nabla^2 V_z + \mu_e \nabla^2 (V_z + d_i \nabla^2 \psi), \tag{13}
\]

whereas the \(z\)-component of its curl reduces to
\[
\left( \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) U = -\frac{d_i}{2} \frac{\tau}{1 + \tau} \left\{ \nabla^2 [\phi, b_z] + [U, b_z] + [\nabla^2 b_z, \phi] \right\} + [\nabla^2 \psi, \psi] \\
+ \mu_i \nabla^2 \left( U + \frac{d_i \tau}{1 + \tau} \nabla^2 b_z \right) + \mu_e \nabla^2 \left( U - \frac{d_i}{1 + \tau} \nabla^2 b_z \right). \tag{14}
\]

Finally, to lowest order, the energy equation, Eq. (6), gives
\[
\left( \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) p_1 = -\frac{\Gamma P_0}{B_0} \nabla \cdot \mathbf{V} + \kappa \nabla^2 p_1. \tag{15}
\]

Eliminating \(\nabla \cdot \mathbf{V}\) between Eqs. (11) and (15), making use of Eq. (12), we obtain
\[
c_\beta^{-2} \left( \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) b_z = [V_z + d_i \nabla^2 \psi, \psi] + \left[ \eta \left( 1 - \frac{3}{2} \frac{\tau}{1 + \tau} \right) + \frac{\kappa}{\beta} \right] \nabla^2 b_z \\
+ d_i \mu_e \nabla^2 \left( U - \frac{d_i}{1 + \tau} \nabla^2 b_z \right). \tag{16}
\]

Here, \(\beta = \frac{\Gamma P_0}{B_0^2}\) is (\(\Gamma\) times) the plasma beta calculated with the “guide-field”, \(B_0\), and \(c_\beta = \sqrt{\beta/(1 + \beta)}\). Note that our ordering scheme does not constrain \(\beta\) to be either much less than or much greater than unity.

Equation (15) implies that \(\nabla \cdot \mathbf{V} \sim O(B_0^{-1})\): \textit{i.e.}, that the flow is \textit{almost} incompressible. Hence, to lowest order, we can write
\[
\mathbf{V} = \nabla \phi \times \mathbf{\hat{z}} + V_z \mathbf{\hat{z}}. \tag{17}
\]

F. Final equations

Let \(d_\beta = c_\beta d_i/\sqrt{1 + \tau}\), \(Z = b_z/c_\beta \sqrt{1 + \tau}\), and \(\bar{V}_z = V_z/\sqrt{1 + \tau}\). Neglecting the bar over \(\bar{V}_z\), our final set of reduced, 2-D, two-fluid, drift-MHD equations takes the form:
\[
\frac{\partial \psi}{\partial t} = [\phi - d_\beta Z, \psi] + \eta J - \frac{\mu_e d_\beta (1 + \tau)}{c_\beta} \nabla^2 [V_z + (d_\beta/c_\beta) J], \tag{18}
\]
\[
\frac{\partial Z}{\partial t} = [\phi, Z] + c_\beta [V_z + (d_\beta/c_\beta) J, \psi] + c_\beta^2 \left[ \eta \left( 1 - \frac{3}{2} \frac{\tau}{1 + \tau} \right) + \frac{\kappa}{\beta} \right] Y
\]
\[ + \mu_e d_\beta \nabla^2(U - d_\beta Y), \quad (19) \]
\[
\frac{\partial U}{\partial t} = [\phi, U] + \frac{d_\beta \tau}{2} \left\{ \nabla^2[\phi, Z] + [U, Z] + [Y, \phi] \right\} + [J, \psi] + \mu_i \nabla^2(U + d_\beta \tau Y) + \mu_e \nabla^2(U - d_\beta Y), \quad (20) \]
\[
\frac{\partial V_z}{\partial t} = [\phi, V_z] + c_\beta [Z, \psi] + \mu_i \nabla^2V_z + \mu_e \nabla^2[V_z + (d_\beta/c_\beta) J]. \quad (21) \]

Here, \( U = \nabla^2\phi \), \( J = \nabla^2\psi \), and \( Y = \nabla^2Z \). The four fields which are evolved in the above equations are the magnetic flux-function, \( \psi \), the (normalized) perturbed \( z \)-directed magnetic field, \( Z \), the \( z \)-directed guiding-center vorticity, \( U \), and the (normalized) \( z \)-directed guiding-center (and ion) fluid velocity, \( V_z \). The (normalized) \( z \)-directed electron fluid velocity is \( V_z + (d_\beta/c_\beta) J \). The quantity \( \phi \) is the guiding-center stream-function. The ion stream-function takes the form \( \phi_i = \phi + d_\beta \tau Z \), whereas the electron stream-function is written \( \phi_e = \phi - d_\beta Z \).

The above equations are “reduced” in the sense that they do not contain the compressible Alfvén wave. However, they do contain the shear-Alfvén wave, the magnetoacoustic wave, the whistler wave, and the kinetic-Alfvén wave. Our equations are similar to the “four-field” equations of Hazeltine, Kotschenreuther, and Morrison,\(^2\) except that they are not limited to small values of \( \beta \).

### III. ISLAND PHYSICS

#### A. Introduction

The aim of this section is to derive expressions determining the phase-velocity and width of a saturated magnetic island (representing the final, nonlinear stage of a tearing instability) from the previously derived set of reduced, 2-D, two-fluid, drift-MHD equations.

Consider a slab plasma which is periodic in the \( y \)-direction with periodicity length \( l \). Let the system be symmetric about \( x = 0 \): i.e., \( \psi(-x, y, t) = \psi(x, y, t), Z(-x, y, t) = -Z(x, y, t), \phi(-x, y, t) = -\phi(x, y, t), \) and \( V_z(-x, y, t) = V_z(x, y, t) \). Consider a quasi-static, constant-\( \psi \) magnetic island, centered on \( x = 0 \). It is convenient to transform to the island rest-frame,
in which $\partial/\partial t \simeq 0$. Suppose that the island is embedded in a plasma with uniform (but different) $y$-directed ion and electron fluid velocities. We are searching for an island solution in which the ion/electron fluid velocities asymptote to these uniform velocities far from the island separatrix.

B. Island geometry

In the immediate vicinity of the island, we can write

$$\psi(x, \theta, t) = -\frac{x^2}{2} + \Psi \cos \theta,$$

where $\theta = y/l$, and $\Psi(t) > 0$ is the reconnected magnetic flux (which is assumed to have a very weak time dependence). As is well-known, the above expression for $\psi$ describes a “cat’s eye” magnetic island of full-width (in the $x$-direction) $W = 4w$, where $w = \sqrt{\Psi}$. The region inside the magnetic separatrix corresponds to $\psi > -\Psi$, the region outside the separatrix corresponds to $\psi < -\Psi$, and the separatrix itself corresponds to $\psi = -\Psi$. The island O- and X-points are located at $(x, \theta) = (0, 0)$, and $(x, \theta) = (0, \pi)$, respectively. Incidentally, we can now conveniently specify the scale length and scale magnetic field-strength introduced in Sect. II C as $a \sim W$ and $B_a \sim B_y^{(0)}(W)$, respectively.

It is helpful to define a flux-surface average operator:

$$\langle f(s, \psi, \theta) \rangle = \int \frac{f(s, \psi, \theta)}{|x|} \frac{d\theta}{2\pi}$$

for $\psi \leq -\Psi$, and

$$\langle f(s, \psi, \theta) \rangle = \int_{-\theta_0}^{\theta_0} \frac{f(s, \psi, \theta) + f(-s, \psi, \theta)}{2|x|} \frac{d\theta}{2\pi}$$

for $\psi > -\Psi$. Here, $s = \text{sgn}(x)$, and $x(s, \psi, \theta_0) = 0$ (with $\pi > \theta_0 > 0$). The most important property of this operator is that

$$\langle [A, \psi] \rangle \equiv 0,$$

for any field $A(s, \psi, \theta)$. 
C. Island equations

The equations governing the quasi-static island [which are derived from Eqs. (18)–(21)] are:

\[
\frac{d\psi}{dt} \cos \theta = [\phi - d_\beta Z, \psi] + \eta \delta J - \frac{\mu_e d_\beta (1 + \tau)}{c_\beta} \nabla^2 [V_z + (d_\beta/c_\beta) \delta J],
\]

\[
0 = [\phi, Z] + c_\beta [V_z + (d_\beta/c_\beta) \delta J, \psi] + c_\beta^2 D Y + \mu_e d_\beta \nabla^2 (U - d_\beta Y),
\]

\[
0 = [\phi, U] + \frac{d_\beta \tau}{2} \left\{ \nabla^2 [\phi, Z] + [U, Z] + [Y, \phi] \right\} + [\delta J, \psi] + \mu_i \nabla^2 (U + d_\beta \tau Y)
\]

\[
+ \mu_e \nabla^2 (U - d_\beta Y),
\]

\[
0 = [\phi, V_z] + c_\beta [Z, \psi] + \mu_i \nabla^2 V_z + \mu_e \nabla^2 [V_z + (d_\beta/c_\beta) \delta J],
\]

where \(\delta J = 1 + \nabla^2 \psi, Y = \nabla^2 Z, U = \nabla^2 \phi\), and

\[D = \eta \left(1 - \frac{3}{2} \frac{\tau}{1 + \tau}\right) + \frac{\kappa}{\beta}.\]

D. Ordering scheme

We adopt the following ordering of terms appearing in Eqs. (26)–(29): \(\psi = \psi^{(0)}, \phi = \phi^{(1)}(s, \psi) + \phi^{(3)}(s, \psi, \theta), Z = Z^{(1)}(s, \psi) + Z^{(3)}(s, \psi, \theta), V_z = V_z^{(2)}(s, \psi, \theta), \delta J = \delta J^{(2)}(s, \psi, \theta)\). Moreover, \(\nabla = \nabla^{(0)}, \tau = \tau^{(0)}, c_\beta = c_\beta^{(0)}, d_\beta = d_\beta^{(0)}, \mu_{i,e} = \mu_{i,e}^{(2)}, \kappa = \kappa^{(2)}, \eta = \eta^{(2)}\), and \(d/dt = d/dt^{(4)}\). Here, the superscript \(^{(i)}\) indicated an \(i\)th order quantity. This ordering, which is completely self-consistent, implies weak \((i.e., strongly\ sub-Alfvénic and sub-magnetooacoustic)\ diamagnetic flows, and very long \((i.e., very much longer than the Alfvén time)\ transport evolution time-scales. According to our scheme, both \(Z\) and \(\phi\) are flux-surface functions, to lowest order. In other words, the lowest order electron and ion stream-functions, \(\phi_e = \phi - d_\beta Z\) and \(\phi_i = \phi + d_\beta \tau Z\), respectively, are flux-surface functions.

To lowest and next lowest orders, Eqs. (26)–(29) yield:

\[
\frac{d^{(4)}\psi}{dt} \cos \theta = [\phi^{(3)} - d_\beta Z^{(3)}, \psi] + \eta^{(2)} \delta J^{(2)} - \frac{\mu_e^{(2)} d_\beta (1 + \tau)}{c_\beta} \nabla^2 [V_z^{(2)} + (d_\beta/c_\beta) \delta J^{(2)}],
\]

\[9\]
0 = c_\beta [V_z^{(2)} + (d_\beta/c_\beta) \delta J^{(2)}, \psi] + c_\beta^2 D^{(2)} Y^{(1)} + \mu_e^{(2)} d_\beta \nabla^2 (U^{(1)} - d_\beta Y^{(1)}),  \quad (32)

0 = -M^{(1)} [U^{(1)}, \psi] + \frac{d_\beta \tau}{2} \left\{ L^{(1)} [U^{(1)}, \psi] + M^{(1)} [Y^{(1)}, \psi] \right\} + [\delta J^{(2)}, \psi]

+ \mu_i^{(2)} \nabla^2 (U^{(1)} + d_\beta \tau Y^{(1)}) + \mu_e^{(2)} \nabla^2 (U^{(1)} - d_\beta Y^{(1)}), \quad (33)

0 = -M^{(1)} [V_z^{(2)}, \psi] + c_\beta [Z^{(3)}, \psi] + \mu_i^{(2)} \nabla^2 V_z^{(2)} + \mu_e^{(2)} \nabla^2 [V_z^{(2)} + (d_\beta/c_\beta) \delta J^{(2)}], \quad (34)

where \( Y^{(1)} = \nabla^2 Z^{(1)} \), and \( U^{(1)} = \nabla^2 \phi^{(1)} \), \( M^{(1)}(s, \psi) = d\phi^{(1)}/d\psi \), and \( L^{(1)}(s, \psi) = dZ^{(1)}/d\psi \).

Here, we have neglected the superscripts on zeroth order quantities, for the sake of clarity.

In the following, we shall neglect all superscripts, except for those on \( \phi^{(3)} \) and \( Z^{(3)} \), for ease of notation.

E. Boundary conditions

It is easily demonstrated that the \( y \)-components of the (lowest order) electron and ion fluid velocities (in the island rest frame) take the form \( V_{e,y} = x(M - d_\beta L) \) and \( V_{i,y} = x(M + d_\beta \tau L) \), respectively. Incidentally, since \( V_{e,y} \) and \( V_{i,y} \) are even functions of \( x \), it follows that \( M(s, \psi) \) and \( L(s, \psi) \) are odd functions. We immediately conclude that \( M(s, \psi) \) and \( L(s, \psi) \) are both zero inside the island separatrix (since it is impossible to have a non-zero odd flux-surface function in this region). Now, we are searching for island solutions for which \( xM \to M_0 \) and \( xL \to L_0 \) as \( |x|/w \to \infty \). In other words, we desire solutions which match to an unperturbed plasma far from the island. If \( V_{e,y}^{(0)} \) and \( V_{i,y}^{(0)} \) are the unperturbed \( y \)-directed electron and ion fluid velocities in the lab. frame, then \( V_{e,y}^{(0)} - V = M_0 - d_\beta L_0 \) and \( V_{i,y}^{(0)} - V = M_0 + d_\beta \tau L_0 \), where \( V \) is the island phase-velocity in the lab. frame. It follows that \( L_0 = (V_{i,y}^{(0)} - V_{e,y}^{(0)})/d_\beta (1 + \tau) \) and \( M_0 = V_{EB,y}^{(0)} - V \), where \( V_{EB,y}^{(0)} \) is the unperturbed plasma \( E \times B \) velocity in the lab. frame. Hence, determining the island phase-velocity is equivalent to determining the value of \( M_0 \).

F. Determination of the flow profiles

Flux-surface averaging Eqs. (32) and (33), we obtain
\[
\langle \nabla^2 U \rangle = -\frac{d_\beta (\mu_i \tau - \mu_e)}{(\mu_i + \mu_e)} \langle \nabla^2 Y \rangle, \quad (35)
\]

and

\[
\delta^2 w^2 \langle \nabla^2 Y \rangle - \langle Y \rangle = 0, \quad (36)
\]

where

\[
\delta = \frac{d_i}{w \sqrt{D} \sqrt{\mu_i \mu_e / (\mu_i + \mu_e)}}, \quad (37)
\]

Assuming that the island is “thin” (i.e., \(w \ll l\)), we can write \(\nabla^2 \approx \partial^2 / \partial x^2\). Hence, Eqs. (35) and (36) yield

\[
M(s, \psi) = -\frac{d_\beta (\mu_i \tau - \mu_e)}{(\mu_i + \mu_e)} L(s, \psi) + F(s, \psi), \quad (38)
\]

where

\[
\frac{d}{d\psi} \left[ \frac{d}{d\psi} \left( \delta^2 w^2 \langle x^4 \rangle \frac{dL}{d\psi} \right) - \langle x^2 \rangle L \right] = 0, \quad (39)
\]

and

\[
\frac{d^2}{d\psi^2} \left( \langle x^4 \rangle \frac{dF}{d\psi} \right) = 0. \quad (40)
\]

We can integrate Eq. (39) once to give

\[
\delta^2 w^2 \frac{d}{d\psi} \left( \langle x^4 \rangle \frac{dL}{d\psi} \right) - \langle x^2 \rangle L = -s L_0. \quad (41)
\]

We can solve Eq. (40), subject to the constraints that \(F\) be continuous, \(F = 0\) inside the separatrix, and \(F \to s F_0\) as \(|x|/w \to \infty\), to give

\[
F(s, \psi) = s F_0 \int_{-\psi}^{\psi} \frac{d\psi}{\langle x^4 \rangle} \int_{-\psi}^{\infty} \frac{d\psi}{\langle x^4 \rangle}, \quad (42)
\]

outside the separatrix. Note that \(x F \to |x| F_0\) as \(|x|/w \to \infty\).

In order to solve Eq. (41), we write \(\hat{\psi} = -\psi / \Psi, \langle \cdots \rangle = \langle \cdots \rangle / w, X = x/w, \) and \(\hat{L} = L/(L_0/w)\). It follows that
\[ \delta^2 \frac{d}{d\psi} \left( \langle X^4 \rangle \frac{d\hat{L}}{d\psi} \right) - \langle X^2 \rangle \hat{L} = -s. \]  

(43)

Suppose that \( \delta \ll 1 \). In this case, \( \hat{L}(s, \hat{\psi}) \) takes the value \( s/\langle X^2 \rangle \) in the region outside the magnetic separatrix, apart from a thin boundary layer on the separatrix itself of width \( \delta w \).

In this layer, the function \( \hat{L}(s, \hat{\psi}) \) makes a smooth transition from its exterior value (which is \( s \pi/4 \) immediately outside the separatrix) to its interior value 0. We can write

\[ \hat{L}(s, \hat{\psi}) = s \left( \frac{1}{\langle X^2 \rangle} + l(y) \right), \]

where \( y = (\hat{\psi} - 1)/\delta \). It follows that

\[ \frac{d^2 l}{dy^2} = \frac{3l}{8} \simeq 0, \]

(45)

since \( \langle X^2 \rangle_{\hat{\psi}=1} = 4/\pi \), and \( \langle X^4 \rangle_{\hat{\psi}=1} = 32/3 \pi \). Hence, the continuous solution to Eq. (41) which satisfies the appropriate boundary conditions is

\[ \hat{L}(s, \hat{\psi}) = s \left[ \frac{1}{\langle X^2 \rangle} - \frac{\pi}{4} \exp \left( -\sqrt{\frac{3}{8}} \frac{\hat{\psi} - 1}{\delta} \right) \right] \]

(46)

in the region outside the separatrix (\( i.e., \hat{\psi} \geq 1 \)). Of course, \( \hat{L}(s, \hat{\psi}) = 0 \) in the region inside the separatrix (\( i.e., \hat{\psi} < 1 \)).

**G. Determination of the island phase-velocity**

Let \( \delta J = \delta J_c + \delta J_s \), where \( \delta J_c \) has the symmetry of \( \cos \theta \), whereas \( \delta J_s \) has the symmetry of \( \sin \theta \). Now, it is easily demonstrated that

\[ \langle \delta J_s \sin \theta \rangle = \frac{l}{\Psi} [\delta J, \psi]. \]

(47)

Hence, it follows from Eq. (33) and (38) that

\[ \langle \delta J_s \sin \theta \rangle = -\frac{l}{\Psi} (\mu_i + \mu_e) \frac{d}{d\psi} \left( \langle x^5 \rangle \frac{d^2 F}{d\psi^2} + 2 \langle x^3 \rangle \frac{dF}{d\psi} - \langle x \rangle F \right). \]

(48)

Now, for an isolated magnetic island which is not interacting electromagnetically with any external structure, such as a resistive wall, the net electromagnetic force acting on the island must be zero. This constraint translates to the well-known requirement that
\[ \int_{-\psi}^{\infty} \langle \delta J_s \sin \theta \rangle d\psi = 0. \] (49)

Using Eq. (48), this requirement reduces to the condition

\[ \lim_{x/w \to \infty} \left( \langle x^5 \rangle \frac{d^2 F}{d\psi^2} + 2 \langle x^3 \rangle \frac{dF}{d\psi} - \langle x \rangle F \right) \propto \lim_{x/w \to \infty} \left[ x^2 \frac{d}{dx} \left( \frac{1}{x} \frac{d(x F)}{dx} \right) \right] \]

\[ = -s F_0 = 0, \] (50)

since \( x F \to |x| F_0 \) as \( |x|/w \to \infty \). Hence, we conclude that \( F_0 = 0 \) \( [i.e., \ F(\psi) = 0, \ everywhere] \) for an isolated magnetic island.

It follows from Eq. (38) that

\[ M(s, \psi) = -\frac{d_\beta (\mu_i \tau - \mu_e)}{\mu_i + \mu_e} L(s, \psi). \] (51)

Hence, \( M_0 = -[d_\beta (\mu_i \tau - \mu_e)/(\mu_i + \mu_e)] L_0 \). Recalling that \( M_0 = V_{EB y}^{(0)} - V, \ V_{iy}^{(0)} = V_{EB y}^{(0)} + d_\beta \tau L_0, \) and \( V_{e_y}^{(0)} = V_{EB y}^{(0)} - d_\beta L_0, \) we obtain the following expression for the island phase-velocity:

\[ V = \frac{\mu_i V_{iy}^{(0)} + \mu_e V_{e_y}^{(0)}}{\mu_i + \mu_e}. \] (52)

In other words, the island phase-velocity is the viscosity weighted mean of the unperturbed ion and electron fluid velocities. Hence, if the ions are far more viscous than the electrons, then the island propagates with the ion fluid. In this case, the ion fluid velocity profile remains largely unaffected by the island, but the electron fluid velocity profile is highly sheared just outside the island separatrix. The opposite is true if the electrons are far more viscous than the ions. This is illustrated in Fig. 1.
FIG. 1. Velocity profiles as functions of $x$, at constant $\theta$, evaluated on a line passing through the island O-point (i.e., at $\theta = 0$) in the island rest frame. The O-point lies at $x = 0$. The island separatrix is indicated by a vertical dotted line. The solid curves show the normalized ion fluid velocity profile: $(V_{i y} - V)/(V_{i y}^{(0)} - V_{e y}^{(0)})$. The short-dashed curves show the normalized electron fluid velocity profile: $(V_{e y} - V)/(V_{i y}^{(0)} - V_{e y}^{(0)})$. The long-dashed curves show the normalized $E \times B$ velocity profile: $(V_{EB y} - V)/(V_{i y}^{(0)} - V_{e y}^{(0)})$. The left-hand panel shows the case of viscous ions: $\mu_e/\mu_i = 0.1$, $\tau = 1.$, and $\delta = 0.2$. The right-hand panel shows the case of viscous electrons: $\mu_i/\mu_e = 0.1$, $\tau = 1.$, and $\delta = 0.2$.

We have now fully specified the ion and electron stream-functions, $\phi_i$ and $\phi_e$, respectively, in the island rest frame. In fact, $\phi_i = 0$ inside the separatrix, and

$$\frac{d\phi_i(s, \hat{\psi})}{d\hat{\psi}} = (V_{i y}^{(0)} - V_{e y}^{(0)}) \frac{\mu_i}{\mu_i + \mu_e} \frac{\hat{L}(s, \hat{\psi})}{w}$$

(53)

outside the separatrix, where the function $\hat{L}(s, \hat{\psi})$ is specified in Eq. (46). Likewise, $\phi_e$ is zero inside the separatrix, and

$$\frac{d\phi_e(s, \hat{\psi})}{d\hat{\psi}} = -(V_{i y}^{(0)} - V_{e y}^{(0)}) \frac{\mu_e}{\mu_i + \mu_e} \frac{\hat{L}(s, \hat{\psi})}{w}$$

(54)

outside the separatrix. Note that the stream-functions and their first derivatives are everywhere continuous, which implies that the ion and electron fluid velocities are everywhere continuous.
H. Determination of the ion polarization current

It follows from Eq. (33) that

$$\delta J_e = \frac{(V - V^{(0)}_{EB_y})(V - V^{(0)}_{iy})}{2} \left( x^2 - \frac{\langle x^2 \rangle}{\langle 1 \rangle} \right) \frac{d}{d\psi} \left( \frac{1}{\langle x^2 \rangle^2} \right) H(\hat{\psi} - 1) + I(s, \psi),$$

(55)

where $I(s, \psi)$ is as yet undetermined. The function $H(\vartheta)$ is zero for $\vartheta < 0$, and unity for $\vartheta \geq 0$. Here, we have made use of the fact that outside the separatrix $L(s, \psi) \simeq s L_0/\langle x^2 \rangle$, and $M(s, \psi) \simeq s M_0/\langle x^2 \rangle$, apart from a thin boundary layer on the separatrix itself. It turns out that we do not need to resolve this boundary layer in order to calculate the total ion polarization current. However, we do have to include the net current flowing in this layer in our calculation of the total current.\textsuperscript{12,14} Flux-surface averaging Eqs. (31) and (34), we obtain

$$\epsilon^2 w^2 \langle \nabla^2 \delta J_e \rangle - \langle \delta J_e \rangle = -\eta^{-1} \frac{d\Psi}{dt} \langle \cos \theta \rangle,$$

(56)

where

$$\epsilon = \frac{d_i}{w} \sqrt{\frac{\mu_i \mu_e}{\mu_i + \mu_e}}.$$  

(57)

Equation (56) implies that

$$\langle \delta J_e \rangle \simeq -\eta^{-1} \frac{d\Psi}{dt} \langle \cos \theta \rangle,$$

(58)

apart from in a thin boundary layer on the separatrix of width $\epsilon w$. Here, we are assuming that $\epsilon \ll 1$. It is easily demonstrated that the deviation of $\langle \delta J_e \rangle$ in the boundary layer from the value given in Eq. (58) makes a negligible contribution to the total ion polarization current. Hence, we shall treat Eq. (58) as if it applied everywhere.

Equations (55) and (58) give

$$\delta J_e = \frac{(V - V^{(0)}_{EB_y})(V - V^{(0)}_{iy})}{2} \left( x^2 - \frac{\langle x^2 \rangle}{\langle 1 \rangle} \right) \frac{d}{d\psi} \left( \frac{1}{\langle x^2 \rangle^2} \right) H(\hat{\psi} - 1) - \eta^{-1} \frac{d\Psi}{dt} \frac{\langle \cos \theta \rangle}{\langle 1 \rangle}.$$  

(59)

Note that this current profile contains no discontinuities or singularities.
The island width evolution equation is obtained by asymptotic matching to the region far from the island.\textsuperscript{3} In fact,
\begin{equation}
\Delta' \Psi = -4 \int_{\Psi}^{\infty} (\delta J_c \cos \theta) \, d\Psi,
\end{equation}
where \( \Delta' \) is the conventional tearing stability index.\textsuperscript{23} It follows from Eqs. (59) and (60) that
\begin{equation}
\Delta' = -\frac{(V - V_{EBy}^{(0)})}{w^3} \frac{(V - V_{iy}^{(0)})}{w^3} \int_{1}^{\infty} \left( \frac{\langle X^4 \rangle}{\langle 1 \rangle} - \frac{\langle X^2 \rangle}{\langle 1 \rangle} \right) \frac{d}{d\Psi} \left( \frac{1}{\langle X^2 \rangle^2} \right) \, d\Psi + \frac{8}{\eta} \frac{dw}{dt} \int_{-1}^{\infty} \frac{\langle \cos \theta \rangle^2}{\langle 1 \rangle} \, d\Psi.
\end{equation}
Performing the flux-surface integrals, whose values are well-known,\textsuperscript{14} we obtain the following island width evolution equation:
\begin{equation}
\frac{0.8227}{\eta} \frac{dW}{dt} = \Delta' + 0.4875 \frac{(V - V_{EBy}^{(0)})}{(W/4)^3} \frac{(V - V_{iy}^{(0)})}{(W/4)^3}.
\end{equation}
Here, \( W = 4w \) is the full island width. The ion polarization current term (the second term on the r.h.s.) is stabilizing when the island phase velocity, \( V \), lies between the unperturbed local \( E \times V \) velocity, \( V_{EBy}^{(0)} \), and the unperturbed local velocity of the ion fluid, \( V_{iy}^{(0)} \).\textsuperscript{16}

IV. SUMMARY AND DISCUSSION

We have developed a novel set of reduced, 2-D, two-fluid, drift-MHD equations. This set of equations takes into account both electron and ion diamagnetism (including the contribution of the ion gyroviscous tensor), as well as the Hall effect and parallel electron compressibility. For the sake of simplicity, the plasma density is assumed to be uniform, and the ion and electron temperatures constant multiples of one another. However, these constraints could easily be relaxed.

Using our equations, we have derived a complete and self-consistent solution for an isolated magnetic island propagating through a slab plasma with uniform but different ion and electron fluid velocities. Our solution is valid provided that the ordering scheme described in Sect. III D holds good, and island width \( W \) is sufficiently large that
\[ W \gg \frac{d_i}{\sqrt{D}} \sqrt{\frac{\mu_i \mu_e}{\mu_i + \mu_e}} \]  
(i.e., \( \delta \ll 1 \)), and

\[ W \gg \frac{d_i}{\sqrt{n}} \sqrt{\frac{\mu_i \mu_e}{\mu_i + \mu_e}} \]  
(i.e., \( \epsilon \ll 1 \)).

Note that the ordering scheme described in Sect. IID implies that \( \omega_s \ll k|| c_s, k|| v_A \) where \( \omega_s \) is a typical diamagnetic frequency, \( c_s \) the sound speed, and \( v_A \) the shear-Alfvén speed. Here, \( k|| \) must be evaluated at the edge of the island. This scheme differs from that adopted in Ref. 16, for which \( k|| c_s \ll \omega_s \). It turns out that our new ordering scheme permits a much less complicated calculation of the flow profiles around the island than that described in Ref. 16.

Within our solution, the ion and electron fluid velocity profiles are uniquely determined in the vicinity of the island [see Fig. 1]. These profiles are everywhere continuous and asymptote to the unperturbed fluid velocities far from the island. Incidentally, the inclusion of electron viscosity in both the Ohm’s law and the plasma equation of motion is key to the determination of continuous velocity profiles.\(^{15}\)

The island propagation velocity is uniquely specified by the condition that there be zero net electromagnetic force acting on the island [see Eq. (52)]. It turns out that the propagation velocity is the viscosity weighted mean of the unperturbed ion and electron fluid velocities. In this paper, we have adopted phenomenological diffusive ion and electron viscosity operators, which are supposed to represent anomalous perpendicular momentum transport due to small-scale plasma turbulence. We have not considered neoclassical viscous forces generated by island toroidal symmetry breaking,\(^{17}\) since such forces are unlikely to dominate the corresponding viscous forces due to turbulent momentum transport unless the island is very wide.

The ion polarization current correction to the Rutherford island width evolution equation is found to be stabilizing when the island propagation velocity lies between the unperturbed
ion fluid velocity and the unperturbed $E \times B$ velocity [see Eq. (62)].\textsuperscript{16} It follows, from our result for the island propagation velocity, that the polarization term is \textit{stabilizing} provided that the anomalous perpendicular ion viscosity \textit{significantly exceeds} the anomalous perpendicular electron viscosity [see Fig. 1, left panel]. Conversely, the polarization term is destabilizing when the electron viscosity significantly exceeds the ion viscosity [see Fig. 1, right panel].\textsuperscript{24} Note, however, that in order for the electron viscosity to exceed the ion viscosity, the electron momentum confinement time would need to be at least a \textit{mass ratio smaller} than the ion momentum confinement time, which does not seem very probable. Hence, we conclude that under normal circumstances the polarization term is \textit{stabilizing}.

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