

Determination of the Non-Ideal Response of a High Temperature Tokamak Plasma to a Static External Magnetic Perturbation via Asymptotic Matching

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Asymptotic matching techniques are used to calculate the response of a high temperature tokamak plasma with a realistic equilibrium to an $n = 1$ static external magnetic perturbation. The plasma is divided into two regions. In the outer region, which comprises most of the plasma, the response is governed by the linearized, marginally-stable, ideal-MHD equations. In the inner region, which is strongly localized around the various rational surfaces within the plasma (where the marginally-stable, ideal-MHD equations becomes singular), non-ideal effects such as resistivity and inertia are taken into account. The recently developed TOMUHAWC code is used to calculate the response in the outer region. The response in the inner region is obtained from Glasser-Greene-Johnson linear layer physics. For the sake of simplicity, the paper focuses on the situation where the plasma at one of the internal rational surfaces is locked to the external perturbation, whereas that at the other surfaces is rotating.

I. INTRODUCTION

Tokamak plasmas are highly sensitive to externally generated, static, helical magnetic perturbations. Such perturbations, which are conventionally termed “error-fields,” are present in all tokamak experiments because of imperfections in magnetic field-coils. An error-field can drive magnetic reconnection in an otherwise tearing-stable plasma, giving rise to the formation of “locked” (i.e., non-rotating) magnetic island chains on so-called “rational” magnetic flux-surfaces within the plasma.¹ Such chains, which are generally known

as “locked modes,” severely degrade global energy confinement,² and often trigger major disruptions.³⁻⁵ Tokamak plasmas are particularly vulnerable to locked modes in their low- β startup phases,³⁻⁸ and also as they approach the ideal β -limit.^{9,10}

The response of a tokamak plasma to an external static magnetic perturbation is generally very different to that predicted by naively superimposing the vacuum perturbation onto the equilibrium magnetic field. A major cause of this difference is the excitation of shielding currents at the various rational surfaces within the plasma. These currents act to suppress driven magnetic reconnection, and must be overcome before significant locked mode formation can occur.^{1,11} Shielding currents are excited by plasma rotation,¹ or by a combination of pressure gradients and the characteristic favorable average magnetic field-line curvature present in tokamak plasmas.¹² The latter mechanism for exciting shielding currents is of particular significance for ITER,¹³ because of the low intrinsic plasma rotation expected in this device. In addition to shielding currents at rational surfaces, an external magnetic perturbation excites currents that are distributed throughout the bulk of the plasma. In a tokamak equilibrium with a realistic aspect-ratio, boundary shape, and pressure, these currents profoundly modify the perturbation.¹⁴ Consequently, cylindrical models, or models that rely on large aspect-ratio, low- β orderings (all of which do a very poor job of modeling the distributed currents), are unable to accurately predict the response of a realistic tokamak equilibrium to a static magnetic perturbation, and often give highly misleading results.¹⁴

There are three main approaches to accurately modeling the response of a realistic tokamak equilibrium to a static external magnetic perturbation. The first approach involves solving the equations of resistive magnetohydrodynamics (resistive-MHD) throughout the whole plasma.^{15,16} Unfortunately, this approach is both time-consuming and highly inefficient—basically, because inertial and resistive effects are only important in the immediate vicinities of the various rational surfaces within the plasma, and the response of the bulk of the plasma to the external perturbation is governed by the equations of marginally-stable, ideal magnetohydrodynamics (ideal-MHD), which are much simpler than the full resistive-MHD equations. The second approach is to solve the marginally-stable, ideal-MHD equations throughout the whole plasma, placing ideal current sheets at the rational surfaces.¹⁷ Such an approach facilitates very rapid calculations, but is of limited accuracy, because it neglects any non-ideal response of the plasma to the external perturbation at the rational surfaces, and this response is often crucially important. The final approach is to use an asymp-

otic matching code—such as the recently developed TOMUHAWC code²³—that solves the marginally-stable, ideal-MHD equations throughout the bulk of the plasma, and asymptotically matches the solutions thus obtained to non-ideal layer (or magnetic island) solutions at the rational surfaces.^{18–23} This approach, which is much more efficient than the first, and is able to capture the non-ideal effects neglected in the second, is adopted in this paper.

II. HOMOGENEOUS TEARING/TWISTING DISPERSION RELATION

Consider a tokamak plasma surrounded by a vacuum region that is bounded by a radially thin, perfectly conducting wall. In the following, all lengths are normalized to the major radius of the plasma magnetic axis, R_0 , all magnetic field-strengths to the vacuum toroidal field-strength at the magnetic axis, B_0 , and all plasma pressures to B_0^2/μ_0 . We employ a right-handed flux coordinate system, r, θ, ϕ . Here, r is a flux-surface label with dimensions of length which is such that $r = 0$ at the magnetic axis, $r = b > 0$ at the plasma boundary, and $r = a > b$ at the wall. Thus, the vacuum corresponds to the region $b < r < a$. Moreover, θ is a “straight” poloidal angle defined such that $\theta = 0$ on the inboard midplane. Finally, ϕ is the geometric toroidal angle. The Jacobian of the coordinate system is (see Sect. A 1)

$$(\nabla r \times \nabla \theta \cdot \nabla \phi)^{-1} = r R^2. \quad (1)$$

(Here, R, ϕ, Z is a conventional right-handed cylindrical coordinate system whose symmetry axis corresponds to the toroidal symmetry axis of the plasma.) The equilibrium magnetic field is written (see Sect. A 2)

$$\mathbf{B}(r, \theta) = \frac{r g}{q} \nabla(\phi - q \theta) \times \nabla r \quad (2)$$

where $q(r)$ is the safety-factor profile, and $g(r)/R$ the toroidal magnetic field-strength.

Consider a global resistive instability whose toroidal mode number is $n > 0$. As is well known, the determination of global resistive stability in a high temperature tokamak plasma can be reduced to an asymptotic matching problem.²⁴ The system is conveniently divided into two regions. In the “outer” region, which comprises most of the plasma, the perturbation is described by the linearized, marginally-stable (i.e., zero inertia), equations of ideal-MHD. However, these equations become singular on rational magnetic flux-surfaces. In the “inner” region, which is strongly localized around the various rational surfaces, non-ideal effects such

as resistivity and inertia become important. The non-ideal layer solution in the vicinity of a given rational surface can generally be separated into independent tearing and twisting (i.e., interchange) parity components.²⁵ Finally, simultaneous asymptotic matching of the layer solutions in the inner region to the ideal-MHD solution in the outer region yields a matrix tearing/twisting dispersion relation.^{18–22,26,27}

Let the m_j , for $j = 1, J$, be the coupled poloidal harmonics included in the stability calculation. The linearized, marginally-stable, ideal-MHD equations that govern the perturbation in the outer region become singular at the various rational surfaces lying within the plasma. These surfaces satisfy $q(r_k) = m_k/n$, for $k = 1, K$, where $0 < r_k < b$, and m_k is one of the m_j . Writing

$$\delta\mathbf{B} \cdot \nabla r = i \sum_{j=1, J} \frac{\psi_j(r)}{r R^2} \exp[i(m_j \theta - n \phi)], \quad (3)$$

where $\delta\mathbf{B}$ is the perturbed magnetic field, the general analytic solution of the linearized, marginally-stable, ideal-MHD equations in the immediate vicinity of the k th rational surface is such that (see Sect. A 5)

$$\psi_k(r) = \Psi_k^\pm F_k |x_k|^{\nu_{Lk}} + \Delta\Psi_k^\pm F_k \operatorname{sgn}(x_k) |x_k|^{\nu_{Sk}} + A_k x_k + \dots, \quad (4)$$

where

$$x_k = \left(\frac{r - r_k}{r_k} \right), \quad (5)$$

$$F_k = \left(\frac{m_k^2 \langle |\nabla r|^2 \rangle + n^2 r^2}{2 \sqrt{D_{Ik}}} \right)_{r_k}^{1/2}, \quad (6)$$

and $\nu_{Lk} = 1/2 - \sqrt{D_{Ik}}$, $\nu_{Sk} = 1/2 + \sqrt{D_{Ik}}$. Here, $\langle \dots \rangle = \oint (\dots) d\theta/2\pi$ is a flux-surface average operator, and D_{Ik} a standard ideal interchange stability parameter evaluated at the surface [see Eq. (A26)].²⁵ Moreover, the plus superscript refers to the region $x_k > 0$, whereas the minus superscript refers to the region $x_k < 0$. It is helpful to define (see Sect. A 5)

$$\Psi_k^e = \frac{1}{2} (\Psi_k^+ + \Psi_k^-), \quad (7)$$

$$\Psi_k^o = \frac{1}{2} (\Psi_k^+ - \Psi_k^-), \quad (8)$$

$$\Delta\Psi_k^e = \Delta\Psi_k^+ + \Delta\Psi_k^-, \quad (9)$$

$$\Delta\Psi_k^o = \Delta\Psi_k^+ - \Delta\Psi_k^-. \quad (10)$$

The complex quantities Ψ_k^e and Ψ_k^o parameterize the amount of magnetic reconnection generated by tearing- and twisting-parity modes, respectively, at the k th rational surface. (Thus, in a completely ideal plasma, $\Psi_k^e = \Psi_k^o = 0$, for all k .) Moreover, the complex quantities

$$\Delta_k^e = \frac{\Delta\Psi_k^e}{\Psi_k^e}, \quad (11)$$

$$\Delta_k^o = \frac{\Delta\Psi_k^o}{\Psi_k^o}, \quad (12)$$

are completely determined by the tearing- and twisting-parity resistive layer solutions, respectively, in the vicinity of the k th rational surface.²⁵

The linearized, marginally-stable, ideal-MHD equations can be solved numerically in the outer region, subject to the constraint that the solution is well behaved at the magnetic axis, and satisfies the physical boundary condition $\delta\mathbf{B} \cdot \nabla r = 0$ at $r = a$.²³ Simultaneous asymptotic matching of this numerical solution to the analytic solution (4) at every rational surface in the plasma yields the homogeneous tearing/twisting dispersion relation (see Sect. A 6)

$$\sum_{k'=1,K} [(E_{kk'}^e - \delta_{kk'} \Delta_{k'}^e) \Psi_{k'}^e + \Gamma_{kk'} \Psi_{k'}^o] = 0, \quad (13)$$

$$\sum_{k'=1,K} [(E_{kk'}^o - \delta_{kk'} \Delta_{k'}^o) \Psi_{k'}^o + \Gamma'_{kk'} \Psi_{k'}^e] = 0, \quad (14)$$

for $k = 1, K$. Here, the elements of the \mathbf{E}^e , \mathbf{E}^o , $\mathbf{\Gamma}$, and $\mathbf{\Gamma}'$ matrices only depend on the ideal-MHD solution in the outer region, and can be calculated, for example, by the TOMUHAWC code.²³ On the other hand, the Δ_k^e and Δ_k^o values depend on the non-ideal layer solutions in the various segments of the inner region. In particular, the elements of the \mathbf{E}^e , \mathbf{E}^o , $\mathbf{\Gamma}$, and $\mathbf{\Gamma}'$ matrices are independent of the complex growth-rate, γ , of the instability, whereas the Δ_k^e and Δ_k^o values are generally functions of γ . Finally, the elements of the \mathbf{E}^e , \mathbf{E}^o , $\mathbf{\Gamma}$, and $\mathbf{\Gamma}'$ matrices satisfy the important symmetry constraints (see Sect. A 6)

$$E_{kk'}^e = E_{k'k}^{e*}, \quad (15)$$

$$E_{kk'}^o = E_{k'k}^{o*}, \quad (16)$$

$$\Gamma'_{kk'} = \Gamma_{k'k}^*, \quad (17)$$

for $k, k' = 1, K$.

It is easily demonstrated that zero net toroidal electromagnetic torque is exerted on the plasma in the outer region as a consequence of tearing or twisting perturbations (see Sect. A 4).²⁰ On the other hand, the net torque exerted on the segment of the inner region centered on the k th rational surface is (see Sect. A 5)

$$\delta T_k = 2 n \pi^2 [\text{Im}(\Delta_k^e) |\Psi_k^e|^2 + \text{Im}(\Delta_k^o) |\Psi_k^o|^2]. \quad (18)$$

It follows from Eqs. (13)–(17) that

$$\sum_{k=1, K} \delta T_k = 0. \quad (19)$$

In other words, zero net toroidal electromagnetic torque is exerted on the plasma as a whole.

III. INHOMOGENEOUS TEARING/TWISTING DISPERSION RELATION

Suppose that the perfectly conducting wall at $r = a$ is subject to a small amplitude, static, helical displacement

$$\xi^x = \frac{\delta r \nabla r}{|\nabla r|^2}, \quad (20)$$

where

$$\delta r(\theta, \phi) = \Xi(\theta) \exp(-i n \phi) = \sum_{j=1, J} \Xi_j \exp[i(m_j \theta - n \phi)]. \quad (21)$$

Thus, the displaced wall is located at

$$r = a + \delta r(\theta, \phi). \quad (22)$$

The response of the plasma to the static magnetic perturbation generated by the wall displacement is governed by the inhomogeneous tearing/twisting dispersion relation, which takes the form (see Sect. A 7)

$$\sum_{k'=1, K} [(E_{kk'}^e - \delta_{kk'} \Delta_{k'}^e) \Psi_{k'}^e + \Gamma_{kk'} \Psi_{k'}^o] = \chi_k^e, \quad (23)$$

$$\sum_{k'=1, K} [(E_{kk'}^o - \delta_{kk'} \Delta_{k'}^o) \Psi_{k'}^o + \Gamma'_{kk'} \Psi_{k'}^e] = \chi_k^o, \quad (24)$$

for $k = 1, K$. Here,

$$\chi_k^{e,o} = \oint \Xi(\theta) \xi_k^{e,o*}(\theta) \frac{d\theta}{2\pi} = \sum_{j=1, J} \Xi_j \xi_{k,j}^{e,o*}, \quad (25)$$

where

$$\xi_k^{e,o}(\theta) = \sum_{j=1,J} \xi_{k,j}^{e,o} \exp(i m_j \theta). \quad (26)$$

As explained in Sect. A 7, the $\xi_{k,j}^{e,o}$ can be calculated from the homogeneous tearing/twisting solutions introduced in Sect. II. Finally, the net toroidal electromagnetic torque exerted by the external perturbation on the segment of the inner region centered on the k th rational surface is (see Sect. A 7)

$$\delta T_k^x = -2 n \pi^2 \text{Im} (\Psi_k^{e*} \chi_k^e + \Psi_k^{o*} \chi_k^o). \quad (27)$$

IV. LINEAR LAYER RESPONSE MODEL

We can make no further progress without adopting a particular model that describes the non-ideal physics governing the segments of the inner region centered on the various rational surfaces within the plasma. Suppose that each segment is governed by Glasser-Greene-Johnson linear resistive-MHD layer physics.²⁵ This model is particularly appropriate to cases in which the plasma is stable to tearing/twisting instabilities, and there is strong shielding of the plasma interior from the external magnetic perturbation (i.e., the external perturbation drives very little magnetic reconnection within the plasma, and, hence, the plasma response at the various rational surfaces is accurately modeled by linear theory).

Let $\rho(r)$, $\eta(r)$, and $\Omega_\phi(r)$ be the plasma mass density, resistivity, and toroidal angular velocity profiles, respectively. The functions

$$\omega_A(r) = \frac{B_0}{R_0} \frac{1}{\sqrt{\mu_0 \rho(r)}}, \quad (28)$$

$$\omega_R(r) = \frac{\eta(r)}{\mu_0 \bar{a}^2}, \quad (29)$$

$$S(r) = \frac{\omega_A}{\omega_R}, \quad (30)$$

where \bar{a} is the wall minor radius (see Sect. V), are the shear-Alfvén frequency, the resistive diffusion rate, and the Lundquist number profiles, respectively. According to Glasser-Greene-Johnson resistive layer theory, we can write²³

$$\Delta_k^{e,o} = S_k^{\mu_k} \tilde{\Delta}_k^{e,o} (\tilde{Q}_k = -i n \tilde{\Omega}_k), \quad (31)$$

where

$$\tilde{\Omega}_k = \frac{\Omega_k}{\omega_{Ak}^{2/3} \omega_{Rk}^{1/3}}, \quad (32)$$

and $\mu_k = (2/3)\sqrt{D_{I_k}}$, $S_k = S(r_k)$, $\Omega_k = \Omega_\phi(r_k)$, $\omega_{Ak} = \omega_A(r_k)$, $\omega_{Rk} = \omega_R(r_k)$. (Here, we have assumed that $\gamma = 0$, because we are dealing with the response of a tearing/twisting stable plasma to a static magnetic perturbation.) Note that the normalized layer quantities $\tilde{\Delta}_k^{e,o}$, which are determined via numerical solution of the Glasser-Greene-Johnson layer equations,^{23,28} are independent of the local Lundquist numbers, S_k .

V. PLASMA EQUILIBRIUM

The up-down symmetric plasma equilibrium whose response to external magnetic perturbations is investigated in this paper is derived from the CHEASE code.²⁹ The unperturbed location of the perfectly conducting wall surrounding the plasma is written parametrically as

$$R = R_c [1 + \bar{a} \cos(\omega + \bar{\delta} \sin \omega)], \quad (33)$$

$$Z = R_c \bar{a} \kappa \sin \omega \quad (34)$$

for $0 \leq \omega \leq 2\pi$. Here, \bar{a} is the wall minor radius, κ the elongation, and $\bar{\delta}$ the triangularity. The parameter R_c is automatically adjusted to ensure that the plasma magnetic axis lies at $R = 1$. The chosen equilibrium profiles are

$$\frac{dP}{d\Psi} = P_0 \begin{cases} (\Psi - \Psi_0)/(\Psi - \Psi_b) & \Psi_0 < \Psi < \Psi_b \\ 0 & \Psi_b \leq \Psi \leq 0 \end{cases}, \quad (35)$$

and

$$g(\Psi) \frac{dg}{d\Psi} = G_0 \begin{cases} [(\Psi - \Psi_0)/(\Psi - \Psi_b)]^\nu & \Psi_0 < \Psi < \Psi_b \\ 0 & \Psi_b \leq \Psi \leq 0 \end{cases}, \quad (36)$$

where

$$\Psi_b = (1 - s_b^2) \Psi_0, \quad (37)$$

and $0 < s_b < 1$. Here, $P(\Psi)$ is the plasma pressure profile,

$$\Psi(r) = - \int_r^a \frac{r g}{q} dr \quad (38)$$

the poloidal magnetic flux, and $\Psi_0 \equiv \Psi(0)$, $\Psi_b \equiv \Psi(b)$. Note that $\Psi(a) = 0$. The parameter G_0 is adjusted such that the central safety-factor, $q_0 \equiv q(0)$, takes a particular value. The

parameter ν is adjusted such that the edge safety-factor, $q_b \equiv q(b)$, takes a particular value. Finally, the parameter P_0 is adjusted such that the normal beta,

$$\beta_N = \frac{\beta(\%) \bar{a}(\text{m}) B_0(\text{T})}{I_p(\text{MA})}, \quad (39)$$

takes a particular value. Here, the total toroidal current, I_p , and total plasma beta, β , are defined in Ref. 29.

VI. EXAMPLE CALCULATION

A. Example Equilibrium

Consider an example plasma equilibrium for which the parameters \bar{a} , κ , and $\bar{\delta}$ are given the JET-like values 0.3, 1.8, and 0.25, respectively. The chosen values of the central safety factor, the edge safety factor, and the normal beta are $q_0 = 1.05$, $q_b = 3.95$, and $\beta_N = 0.25$, respectively. Finally, the boundary parameter s_b is given the value 0.95. It follows that $a = 0.4057$ and $b = 0.3557$. Furthermore, there are two $n = 1$ rational surfaces in the plasma: the $q = 2$ (or $k = 1$) surface at $r/b = 0.7283$, and the $q = 3$ (or $k = 2$) surface at $r/b = 0.9075$. The equilibrium magnetic flux-surfaces, as well as the location of the wall and the plasma boundary, are shown in Fig. 1. The safety-factor profile is shown in Fig. 2.

B. Calculation of $n = 1$ Shielding Factors

The inhomogeneous tearing/twisting dispersion relation (23)–(24) can be written in the general form [see Eq. (31)]

$$\sum_{k'=1, K} \left[(E_{kk'}^e - \delta_{kk'} S_{k'}^{\mu_{k'}} \tilde{\Delta}_{k'}^e) \Psi_{k'}^e + \Gamma_{kk'} \Psi_{k'}^o \right] = \chi_k^e, \quad (40)$$

$$\sum_{k'=1, K} \left[(E_{kk'}^o - \delta_{kk'} S_{k'}^{\mu_{k'}} \tilde{\Delta}_{k'}^o) \Psi_{k'}^o + \Gamma'_{kk'} \Psi_{k'}^e \right] = \chi_k^o, \quad (41)$$

for $k = 1, K$. In a high temperature tokamak plasma, the Lundquist numbers, S_k , are typically very much greater than unity, and also much greater than the $|E_{kk'}^{e,o}|$, $|\Gamma_{kk'}|$, $|\Gamma'_{kk'}|$, and $|\tilde{\Delta}_k^{e,o}|$. It follows that the external perturbation drives negligible magnetic reconnection inside the plasma (i.e., the $|\Psi_k^{e,o}|$ are very much less than the $|\chi_k^{e,o}|$), except when one, or more, of the $|\tilde{\Delta}_k^e|$ or the $|\tilde{\Delta}_k^o|$ is small compared to unity.

Figures 3 and 4 show the variation of the $n = 1$ twisting- and tearing-parity layer response functions at the $q = 2$ surface— $\tilde{\Delta}_1^o$ and $\tilde{\Delta}_1^e$, respectively—with the normalized local plasma toroidal angular velocity, $\tilde{\Omega}_1$, calculated (via the finite-difference method described in Ref. 28) for the example plasma equilibrium illustrated in Figs. 1 and 2. The variation of the corresponding response functions at the $q = 3$ surface with the local toroidal angular velocity is analogous.

It can be seen, from Fig. 3, that the magnitude of the twisting-parity response function, $\tilde{\Delta}_1^o$, is always much greater than unity. This turns out to be a general result. Thus, we deduce that, in general, $|\tilde{\Delta}_k^o| \gg 1$ for $k = 1, K$. This implies, from (41), that the Ψ_k^o are all negligibly small: i.e., the twisting-parity responses of the various segments of the inner region to the external perturbation are negligible. In this situation, the inhomogeneous dispersion relation, (40) and (41), simplifies to give

$$\sum_{k'=1, K} (E_{kk'}^e - \delta_{kk'} S_{k'}^{\mu_{k'}} \tilde{\Delta}_{k'}^e) \Psi_{k'}^e \simeq \chi_k^e, \quad (42)$$

for $k = 1, K$.

It can be seen, from Fig. 4, that the magnitude of the tearing-parity response function, $\tilde{\Delta}_1^e$, is small compared to unity when the magnitude of the normalized local toroidal angular velocity, $\tilde{\Omega}_1$, is of order unity. On the other hand, $|\tilde{\Delta}_1^e| \gg 1$ when $|\tilde{\Omega}_1| \gg 1$. This also turns out to be a general result. Thus, we deduce that, in general, $|\tilde{\Delta}_k^e| \gg 1$ when $|\tilde{\Omega}_k| \gg 1$, for $k = 1, K$. This implies, from (42), that Ψ_k^e is negligibly small when $|\tilde{\Omega}_k| \gg 1$. In the following, we shall suppose that the plasma toroidal angular velocity profile is sufficiently sheared that only one of the $|\tilde{\Omega}_k|$ is comparable with unity, and the others are all much greater than unity. Let the surface in question be the k th surface. Thus, $\Psi_{k' \neq k}^e = 0$, and the inhomogeneous dispersion relation (42) further simplifies to give

$$\Psi_k^e \simeq \frac{\chi_k^e}{E_{kk}^e - S_k^{\mu_k} \tilde{\Delta}_k^e}. \quad (43)$$

According to Eq. (27), the toroidal electromagnetic torque exerted by the external perturbation on the plasma in the immediate vicinity of the k th rational surface takes the form

$$\delta T_k^x \simeq 2 n \pi^2 \frac{S_k^{\mu_k} \text{Im}(\tilde{\Delta}_k^e) |\chi_k^e|^2}{|E_{kk}^e - S_k^{\mu_k} \tilde{\Delta}_k^e|^2}. \quad (44)$$

It is evident, from Fig. 4, that this torque acts to adjust the local toroidal angular velocity, $\tilde{\Omega}_k$, such that $\text{Im}(\tilde{\Delta}_k) = 0$, at which point $|\tilde{\Delta}_k|$ attains a minimum value: i.e., the shielding

of the k th rational surface from the external perturbation is minimized. Let us suppose that this process, which we shall term the “locking” of the k th rational surface to the external perturbation, has taken place. Let $\tilde{\Omega}_{k_c}$ be the critical local normalized toroidal angular velocity at which $\text{Im}(\tilde{\Delta}_k) = 0$, and let $\tilde{\Delta}_{k_c}$ be the corresponding value of $\tilde{\Delta}_k$. Note that $\tilde{\Delta}_{k_c}$ is real and positive.

Suppose that the tearing-parity mode that only reconnects magnetic flux at the k th rational surface is intrinsically unstable: i.e., $E_{kk}^e > 0$.^{20,23} In this situation, Eq. (43) becomes

$$\Psi_k^e \simeq -\frac{\chi_k^e}{|E_{kk}^e|} \Sigma_k(S_k), \quad (45)$$

where

$$\Sigma_k(S_k) = \frac{1}{(S_k/S_{k_c})^{\mu_k} - 1}, \quad (46)$$

and

$$S_{k_c} = \left(\frac{|E_{kk}^e|}{\tilde{\Delta}_{k_c}^e} \right)^{1/\mu_k} \quad (47)$$

is the critical local Lundquist number above which the mode in question is stabilized by the Glasser effect (i.e., by the stabilizing influence of the characteristic favorable average magnetic field-line curvature present in tokamak plasmas).^{12,23,25} In expression (45), $|\chi_k^e|/|E_{kk}^e|$ is the magnitude of the tearing-parity magnetic flux that would be driven at the k th rational surface by the external perturbation in the absence of any shielding at the surface (i.e., when $\tilde{\Delta}_k^e = 0$). Thus, $\Sigma_k(S_k)$ can be interpreted as the “shielding factor” at the k th surface: i.e., the factor by which the flux driven at the surface is reduced by the layer response. Obviously, expression (46) is only valid when $S_k > S_{k_c}$ (i.e., when the plasma is tearing stable). In the high Lundquist limit, $S_k \gg S_{k_c}$, the magnitude of the shielding factor becomes very much less than unity. This implies that, in a high temperature tokamak plasma, there can be strong shielding of a given rational surface from a resonant external magnetic perturbation, even when the surface in question is locked to the perturbation. This shielding is due to the Glasser effect.¹² As $S_k \rightarrow S_{k_c}$, the shielding factor grows in magnitude, eventually becoming infinite at the marginal stability point, $S_k = S_{k_c}$. In other words, the shielding breaks down as the tearing-parity mode that reconnects flux at the k th rational surface approaches marginal stability. In fact, when the mode is close to marginal stability, the magnitude of the shielding factor becomes greater than unity, indicating that the layer response actually amplifies the reconnected magnetic flux driven at the k th rational surface.

Suppose that the tearing-parity mode that only reconnects magnetic flux at the k th rational surface is intrinsically stable: i.e., $E_{kk}^e < 0$.^{20,23} In this situation, Eq. (43) becomes

$$\Psi_k^e \simeq -\frac{\chi_k^e}{|E_{kk}^e|} \Sigma_k(S_k), \quad (48)$$

where

$$\Sigma_k(S_k) = \frac{1}{(S_k/S_{kc})^{\mu_k} + 1}, \quad (49)$$

and

$$S_{kc} = \left(\frac{|E_{kk}^e|}{\tilde{\Delta}_{kc}^e} \right)^{1/\mu_k}. \quad (50)$$

In the high Lundquist limit, $S_k \gg S_{kc}$, the magnitude of the shielding factor, $\Sigma_k(S_k)$, is very much less than unity: i.e., there is strong shielding. On the other hand, in the low Lundquist number limit, $S_k \ll S_{kc}$, the magnitude of the shielding factor is unity: i.e., there is no shielding. Hence, we can interpret S_{kc} as the critical Lundquist number at the k th rational surface above which there is strong shielding due to the Glasser effect.

For the example equilibrium illustrated in Figs. 1 and 2, the TOMUHAWC code²³ yields $E_{11}^e = 3.678$ and $E_{22}^e = -6.421$. We also have $\mu_1 = 0.338$, $\tilde{\Delta}_{1c}^e = 2.535 \times 10^{-2}$, $\tilde{\Omega}_{1c} = \pm 4.864 \times 10^{-2}$, $\mu_2 = 0.335$, $\tilde{\Delta}_{2c}^e = 1.289 \times 10^{-2}$, and $\tilde{\Omega}_{2c} = \pm 1.301 \times 10^{-2}$. Thus, $S_{1c} = 2.47 \times 10^6$ and $S_{2c} = 1.14 \times 10^8$. It follows that the shielding factor at the $q = 2$ surface (assuming that this surface is locked, and the plasma at the $q = 3$ surface is rapidly rotating: i.e., $|\tilde{\Omega}_2| \gg |\tilde{\Omega}_{2c}|$) is

$$\Sigma_1(S_1) = \frac{1}{(S_1/2.47 \times 10^6)^{0.338} - 1}. \quad (51)$$

(Obviously, if the $q = 2$ surface is not locked then the shielding factor will be smaller: i.e., there will be more shielding. Thus, the above expression specifies the least possible amount of shielding at the $q = 2$ surface.) On the other hand, the shielding factor at the $q = 3$ surface (assuming that this surface is locked, and the plasma at the $q = 2$ surface is rapidly rotating: i.e., $|\tilde{\Omega}_1| \gg |\tilde{\Omega}_{1c}|$) is

$$F_2(S_2) = \frac{1}{(S_2/1.14 \times 10^8)^{0.335} + 1}. \quad (52)$$

C. Calculation of Optimal $n = 1$ Wall Displacements

According to Eq. (25), for a wall displacement of fixed amplitude, the magnitude of the parameter χ_k^e is maximized when $\Xi_j \propto \xi_{k,j}^e$. Hence, according to Eq. (43), for a fixed

plasma response, the tearing-parity magnetic flux driven at the k th rational surface is also maximized. We conclude that the Fourier harmonics of the optimal wall displacement for driving tearing-parity magnetic reconnection at the k th rational surface can be written

$$\Xi_j = -\delta \hat{\xi}_{k,j}^e, \quad (53)$$

where

$$\hat{\xi}_{k,j}^e = \frac{\xi_{k,j}^e}{\|\xi_k^e\|}, \quad (54)$$

and

$$\|\xi_k^e\| = \left(\sum_{j=1,J} |\xi_{k,j}^e|^2 \right)^{1/2}. \quad (55)$$

Here, δ is the mean wall displacement (in r). Thus, the optimally displaced wall lies at

$$r = a - \text{Re} \sum_{j=1,J} \delta \hat{\xi}_{k,j}^e \exp [i(m_j \theta - n \phi)]. \quad (56)$$

If the $\hat{\xi}_{k,j}^e$ are real (which is the case when the plasma equilibrium is up-down symmetric) then the optimally displaced wall lies at

$$r = a \pm \delta C_k(\theta) \quad (57)$$

for $n\phi = 0, \pi$, and at

$$r = a \pm \delta S_k(\theta) \quad (58)$$

for $n\phi = \pi/2, 3\pi/2$. Here,

$$C_k(\theta) = \sum_{j=1,J} \hat{\xi}_{k,j} \cos(m_j \theta), \quad (59)$$

$$S_k(\theta) = \sum_{j=1,J} \hat{\xi}_{k,j} \sin(m_j \theta). \quad (60)$$

Finally, from (45) and (48), if the wall is optimally displaced then the tearing-parity magnetic flux driven at the k th rational surface is

$$\Psi_k^e = \delta \frac{\|\xi_k^e\|}{|E_{kk}^e|} F_k(S_k). \quad (61)$$

Figures 5 and 6 show the optimal $n = 1$ wall displacement for driving tearing-parity magnetic flux at the $q = 2$ surface, calculated for our example equilibrium by the TOMUHAWC code.²³ It can be seen that the optimal displacement takes the form of a distorted $m = 3/n =$

1 helical oscillation in the geometric poloidal and toroidal angles. Figures 7 and 8 show the optimal $n = 1$ wall displacement for driving tearing-parity magnetic flux at the $q = 3$ surface, calculated for the same equilibrium by the TOMUHAWC code.²³ In this case, the optimal displacement takes the form of a distorted $m = 4/n = 1$ helical oscillation in the geometric poloidal and toroidal angles. In both cases, the distortion is such that the amplitude of the oscillation is particularly large above and below the plasma, and particularly small on the inboard side.³⁰ This indicates that the $q = 2$ and $q = 3$ surfaces respond preferentially to $n = 1$ wall displacements above and below the plasma, and are relatively insensitive to wall displacements on the inboard side.

VII. BETA SCAN

Consider a series of plasma equilibria, characterized by $\bar{a} = 0.3$, $\kappa = 1.8$, $s_b = 0.95$, $\bar{\delta} = 0.25$, $q_0 = 1.05$, and $q_b = 3.95$, for which the normal beta, β_N , varies between zero and the $n = 1$ beta limit, $\beta_{Nc} = 2.955$. The example equilibrium considered in Sect. VI is a member of this series corresponding to $\beta_N = 0.25$.

Figure 9 shows the shielding factor for tearing-parity magnetic flux driven at the $q = 2$ surface by an $n = 1$ wall displacement, calculated by the TOMUHAWC code for the series of equilibria in question. Figure 10 shows the corresponding ratio of the flux driven at the $q = 2$ surface to the mean wall displacement, when the wall displacement is optimal. It can be seen that strong shielding (i.e., $F_1 \ll 1$) is achieved at intermediate β_N values when the local Lundquist number is relatively high (i.e., $S_1 \gg 10^6$), but that the shielding is always weak (i.e., $F_1 \sim 1$) when the local Lundquist number is relatively low (i.e., $S_1 \sim 10^6$). Furthermore, the shielding breaks down completely, and actually becomes amplification (i.e., $F_1 \gg 1$), both in the low- β limit, $\beta_N \rightarrow 0$, and in the high- β limit, $\beta_N \rightarrow \beta_{Nc}$. The shielding breaks down in the low- β limit because the Glasser effect (which depends on the local pressure gradient) becomes too feeble to stabilize the $n = 1$ tearing mode that reconnects flux at the $q = 2$ surface (which is intrinsically unstable: i.e., $E_{11}^e > 0$). The shielding breaks down in the high- β limit because the tearing stability index E_{11}^e tends to infinity as $\beta_N \rightarrow \beta_{Nc}$.³¹ Figures 9 and 10 suggest that the well-known susceptibility of low- β startup plasmas to highly amplified locked modes³⁻⁸ may be due to the breakdown of Glasser-effect shielding of an intrinsically unstable tearing mode from a resonant error-

field, as the plasma pressure becomes too low. The figures also account for the common appearance of highly amplified locked modes in tokamak plasmas that approach the ideal β -limit.^{9,10}

Figure 11 shows the shielding factor for tearing-parity magnetic flux driven at the $q = 3$ surface by an $n = 1$ wall displacement, calculated by the TOMUHAWC code for the series of equilibria in question. Figure 12 shows the corresponding ratio of the flux driven at the $q = 3$ surface to the mean wall displacement, when the wall displacement is optimal. It can be seen that, although the shielding fails (i.e., $F_2 \rightarrow 1$) as $\beta_N \rightarrow 0$, there is no amplification of the driven magnetic flux in this limit, because the $n = 1$ tearing mode that reconnects magnetic flux at the $q = 3$ surface is intrinsically stable (i.e., $E_{22}^e < 0$) at low- β . On the other hand, there is again strong amplification of the flux driven at the $q = 3$ surface when the plasma approaches the ideal β -limit.

Figures 13 and 14 show the optimal $n = 1$ wall displacement for driving tearing-parity magnetic flux at the $q = 2$ surface, calculated by the TOMUHAWC code when the plasma lies very close to the ideal β -limit. Likewise, Figs. 15 and 16 show the optimal $n = 1$ wall displacement for driving tearing-parity magnetic flux at the $q = 3$ surface. In both cases, the optimal displacement takes the form of a distorted $m = 3/n = 1$ helical oscillation in the geometric poloidal and toroidal angles. The distortion is such that the amplitude of the oscillation is particularly large above and below the plasma, and on the outboard side, but is virtually zero on the inboard side.³⁰ This indicates that, close to the β -limit, the $q = 2$ and $q = 3$ surfaces respond preferentially to $n = 1$ wall displacements above and below the plasma, and on the outboard side, but are completely insensitive to wall displacements on the inboard side.³⁰

Note that Figs. 13 and 15 are virtually identical to one another. The same can be said for Figs. 14 and 16. This suggests that, when the plasma is close to the ideal β -limit, the optimal $n = 1$ wall displacement for driving tearing-parity magnetic flux at the $q = 2$ surface is very similar in shape to the optimal displacement for driving flux at the $q = 3$ surface. In fact, we can measure this similarity by defining the correlation factor,

$$\alpha_{12} = \left| \sum_{j=1, J} \hat{\xi}_{1j} \hat{\xi}_{2j} \right|. \quad (62)$$

Thus, if $\alpha_{12} = 0$ then there is no similarity between the two shapes, whereas if $\alpha_{12} = 1$ then the two shapes are identical. The correlation factor, calculated by the TOMUHAWC code for

the series of plasma equilibria in question, is shown in Fig. 17. It can be seen that at low- β the correlation factor is relatively small (i.e., $0 < \alpha_{12} \ll 1$), indicating that in this limit the optimal $n = 1$ wall displacement for driving flux at the $q = 2$ surface has a quite different shape to that for driving flux at the $q = 3$ surface. On the other hand, the correlation factor approaches unity as the ideal β -limit is approached, indicating that the optimal $n = 1$ wall displacement shape for driving flux at the $q = 2$ surface becomes identical to that for driving flux at the $q = 3$ surface.

VIII. SUMMARY AND DISCUSSION

This paper discusses the use of asymptotic matching techniques to determine the response of a high temperature tokamak plasma with a realistic equilibrium to a static external magnetic perturbation. The plasma is divided into two regions. In the outer region, which comprises most of the plasma, the response is governed by the linearized, marginally-stable, ideal-MHD equations. In the inner region, which is strongly localized around the various rational surfaces within the plasma (where the marginally-stable, ideal-MHD equations becomes singular), non-ideal effects such as resistivity and inertia are taken into account. In this paper, we use the recently developed TOMUHAWC code²³ to calculate the response in the outer region. The response in the inner region is obtained from Glasser-Greene-Johnson linear layer physics.²⁵ For the sake of simplicity, we have focused on the situation where the plasma at one of the internal rational surfaces is locked to the external perturbation, whereas that at the other surfaces is rotating.

In agreement with Liu, et al.,¹² we find that, under certain circumstances, the so-called Glasser effect²⁵ leads to strong shielding of the plasma in the vicinity of an internal rational surface from a static external resonant magnetic perturbation. This is true even in the limit in which the plasma at the rational surface is locked to the perturbation. Such shielding breaks down at low- β , because the Glasser effect becomes too weak. The screening also breaks down as the ideal β -limit is approached, because the plasma becomes too unstable.

By supposing that the static external magnetic perturbation is generated by a small amplitude helical displacement in a perfectly conducting wall surrounding (but, not, necessarily touching) the plasma, it is possible to determine the optimal $n = 1$ wall displacement for driving magnetic reconnection at a given rational surface within the plasma. At low- β , the

optimum displacements for driving magnetic reconnection at different rational surfaces are quite distinct from one another. On the other hand, as the ideal β -limit is approached, the optimum displacements all become identical.

ACKNOWLEDGEMENTS

This research was funded by the U.S. Department of Energy under contract DE-FG02-04ER-54742.

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Appendix A: Solution of Linearized Marginally-Stable Ideal-MHD Equations

1. Normalization and Coordinates

All lengths are normalized to the major radius of the plasma magnetic axis, R_0 , all magnetic field-strengths to the vacuum toroidal field-strength at the magnetic axis, B_0 , and all plasma pressures to B_0^2/μ_0 .

Let R, ϕ, Z be right-handed cylindrical coordinates whose symmetry axis corresponds to the toroidal symmetry axis of the plasma. The Jacobian for these coordinates is

$$(\nabla R \times \nabla \phi \cdot \nabla Z)^{-1} = R. \quad (\text{A1})$$

Let r, θ, ϕ be right-handed flux coordinates whose Jacobian is^{18,20}

$$\mathcal{J}(r, \theta) \equiv (\nabla r \times \nabla \theta \cdot \nabla \phi)^{-1} = r R^2. \quad (\text{A2})$$

Here, r is a flux-surface label with dimensions of length. Furthermore, θ is a straight poloidal angle. Let $r = 0$ correspond to the magnetic axis, and $\theta = 0$ to the inboard midplane.

2. Plasma Equilibrium

Consider an axisymmetric tokamak plasma equilibrium. The magnetic field is written^{18,20}

$$\mathbf{B}(r, \theta) = f(r) \nabla \phi \times \nabla r + g(r) \nabla \phi = f \nabla(\phi - q\theta) \times \nabla r, \quad (\text{A3})$$

where

$$q(r) = \frac{r g}{f} \quad (\text{A4})$$

is the safety-factor profile.

Equilibrium force balance yields the Grad-Shafranov equation,^{18,20}

$$\frac{1}{r} \frac{\partial}{\partial r} (r f |\nabla r|^2) + \frac{1}{r} \frac{\partial}{\partial \theta} (r f \nabla r \cdot \nabla \theta) + \frac{g g'}{f} + \left(\frac{R}{R_0} \right)^2 \frac{P'}{f} = 0, \quad (\text{A5})$$

where $P(r)$ is the plasma pressure profile, and $' \equiv d/dr$.

3. Governing Equations

Consider a small perturbation to the previously described plasma equilibrium. The system is conveniently divided into an outer region and an inner region.²⁴ The outer region comprises all of the plasma, and any surrounding vacuum, except a number of radially thin layers centered on the various rational surfaces. The inner region consists of the aforementioned layers. The perturbation in the outer region is governed by linearized, marginally-stable, ideal-MHD, whereas that in the inner region is governed by either linear or nonlinear resistive-MHD.²⁴ The overall solution is constructed by asymptotically matching the ideal-MHD solution in the outer region to the resistive-MHD solutions in the various segments of the inner region.

The linearized, marginally-stable, ideal-MHD equations that govern the perturbation in the outer region are²⁴

$$\delta\mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}), \quad (\text{A6})$$

$$\nabla\delta P = \delta\mathbf{J} \times \mathbf{B} + \mathbf{J} \times \delta\mathbf{B}, \quad (\text{A7})$$

$$\delta\mathbf{J} = \nabla \times \delta\mathbf{B}, \quad (\text{A8})$$

$$\delta P = -\boldsymbol{\xi} \cdot \nabla P. \quad (\text{A9})$$

Here, $\mathbf{J} = \nabla \times \mathbf{B}$ is the equilibrium current density. Moreover, $\boldsymbol{\xi}$, $\delta\mathbf{B}$, $\delta\mathbf{J}$, and δP are the plasma displacement, perturbed magnetic field, perturbed current density, and perturbed pressure, respectively.

4. Fourier Transformed Equations

Let

$$f \boldsymbol{\xi} \cdot \nabla r = \sum_{j=1, J} \frac{\psi_j(r)}{m_j - nq} \exp [i (m_j \theta - n \phi)], \quad (\text{A10})$$

$$R^2 \delta\mathbf{B} \cdot \nabla \phi = \sum_{j=i, J} \left[\frac{n Z_j(r) + \lambda_j \psi_j(r)}{m_j - nq} \right] \exp [i (m_j \theta - n \phi)], \quad (\text{A11})$$

where

$$\lambda_j = \frac{-m_j (m_j - nq) (g'/f) \langle |\nabla r|^{-2} \rangle + n m_j (r P'/f^2) \langle |\nabla r|^{-2} R^2 \rangle}{m_j^2 \langle |\nabla r|^{-2} \rangle + n^2 r^2}. \quad (\text{A12})$$

Here, the m_j , for $j = 1, J$, are the coupled poloidal harmonics included in the calculation. Moreover, $n > 0$ is the toroidal mode number of the perturbation, and $\langle \dots \rangle \equiv \oint (\dots) d\theta/2\pi$ is a flux-surface average operator. After considerable algebra, Eqs. (A6)–(A9) reduce to^{18,20}

$$r \frac{d\psi_j}{dr} = \sum_{j'=1, J} \frac{L_{jj'} Z_{j'} + M_{jj'} \psi_{j'}}{m_{j'} - nq}, \quad (\text{A13})$$

$$(m_j - nq) r \frac{d}{dr} \left(\frac{Z_j}{m_j - nq} \right) = \sum_{j'=1, J} \frac{N_{jj'} Z_{j'} + P_{jj'} \psi_{j'}}{m_{j'} - nq}, \quad (\text{A14})$$

for $j = 1, J$, where the coupling matrices $L_{jj'}$, $M_{jj'}$, $N_{jj'}$, and $P_{jj'}$ are specified in Ref. 23. These matrices have the following important symmetry properties:

$$L_{j'j} = L_{jj'}^*, \quad (\text{A15})$$

$$M_{j'j} = -N_{jj'}^*, \quad (\text{A16})$$

$$N_{j'j} = -M_{jj'}^*, \quad (\text{A17})$$

$$P_{j'j} = P_{jj'}^*, \quad (\text{A18})$$

for $j, j' = 1, J$. Thus, it follows from Eqs. (A13) and (A14) that

$$r \frac{d}{dr} \left(\sum_{j=1, J} \frac{Z_j^* \psi_j - \psi_j^* Z_j}{m_j - nq} \right) = 0. \quad (\text{A19})$$

The net toroidal electromagnetic torque acting on the region lying within that equilibrium magnetic flux-surface whose label is r takes the form^{20,23}

$$T_\phi(r) = \int_0^r \oint \oint R^2 \nabla \phi \cdot (\delta \mathbf{J} \times \delta \mathbf{B}) \mathcal{J} dr d\theta d\phi, \quad (\text{A20})$$

which can be shown to reduce to

$$T_\phi(r) = n \pi^2 i \sum_{j=1, J} \frac{Z_j^* \psi_j - \psi_j^* Z_j}{m_j - nq}. \quad (\text{A21})$$

Hence, we deduce from (A19) that

$$\frac{dT_\phi}{dr} = 0 \quad (\text{A22})$$

in the outer region. In other words, zero net electromagnetic torque is exerted on the plasma in the outer region.

5. Behavior in Vicinity of Rational Surface

Let there be K rational surfaces in the plasma. Suppose that the k th surface has the flux-surface label r_k , and the resonant poloidal mode number m_k , where $q(r_k) = m_k/n$, for $k = 1, K$. The general analytic solution of the linearized, marginally-stable, ideal-MHD equations in the immediate vicinity of the k th rational surface is such that^{20,23}

$$\psi_k(r) = \Psi_k^\pm F_k |x_k|^{\nu_{Lk}} + \Delta\Psi_k^\pm F_k \operatorname{sgn}(x_k) |x_k|^{\nu_{Sk}} + A_k x_k + \dots, \quad (\text{A23})$$

where

$$x_k = \left(\frac{r - r_k}{r_k} \right), \quad (\text{A24})$$

$$F_k = \left(\frac{m_k^2 \langle |\nabla r|^2 \rangle + n^2 r^2}{2 \sqrt{D_{Ik}}} \right)_{r_k}^{1/2}, \quad (\text{A25})$$

and $\nu_{Lk} = 1/2 - \sqrt{D_{Ik}}$, $\nu_{Sk} = 1/2 + \sqrt{D_{Ik}}$. Here,

$$D_{Ik} = \frac{1}{4} - \frac{r P'}{f^2 s^2} \left\{ \left(\langle |\nabla r|^{-2} \rangle + \frac{r^2}{q^2} \right) \left[-f \frac{d}{dr} \left(\frac{r}{f} \langle R^2 \rangle \right) + \frac{r P'}{f^2} \langle |\nabla r|^{-2} R^4 \rangle \right] - \frac{r P'}{f^2} \langle |\nabla r|^{-2} R^2 \rangle^2 + s \langle |\nabla r|^{-2} R^2 \rangle \right\}, \quad (\text{A26})$$

where $s = r q'/q$, is a standard ideal interchange stability parameter evaluated at the k th rational surface.^{23,25} Moreover, the plus superscript refers to the region $x_k > 0$, whereas the minus superscript refers to the region $x_k < 0$.

It is helpful to define

$$\Psi_k^e = \frac{1}{2} (\Psi_k^+ + \Psi_k^-), \quad (\text{A27})$$

$$\Psi_k^o = \frac{1}{2} (\Psi_k^+ - \Psi_k^-), \quad (\text{A28})$$

$$\Delta\Psi_k^e = \Delta\Psi_k^+ + \Delta\Psi_k^-, \quad (\text{A29})$$

$$\Delta\Psi_k^o = \Delta\Psi_k^+ - \Delta\Psi_k^-. \quad (\text{A30})$$

The complex quantities Ψ_k^e and Ψ_k^o parameterize the amount of magnetic reconnection generated by tearing- and twisting-parity modes, respectively, at the k th rational surface. (Thus, in a completely ideal plasma, $\Psi_k^e = \Psi_k^o = 0$, for all k .) Furthermore, the complex quantities

$$\Delta_k^e = \frac{\Delta\Psi_k^e}{\Psi_k^e}, \quad (\text{A31})$$

$$\Delta_k^o = \frac{\Delta\Psi_k^o}{\Psi_k^o}, \quad (\text{A32})$$

are completely determined by the tearing- and twisting-parity resistive layer solutions, respectively, in the vicinity of the k th rational surface,²³ Finally, the net torque electromagnetic torque exerted on the segment of the inner region centered on the k th rational surface is^{20,23}

$$\delta T_k = 2 n \pi^2 \text{Im} (\Delta \Psi_k^e \Psi_k^{e*} + \Delta \Psi_k^o \Psi_k^{o*}). \quad (\text{A33})$$

6. Derivation of Homogeneous Dispersion Relation

Let $\mathbf{y}(r)$ represent the $2J$ -dimensional vector of the $\psi_j(r)$ and $Z_j(r)$ functions that satisfy the outer equations, (A13) and (A14).

Suppose that the plasma boundary corresponds to the flux-surface $r = b$, and that the plasma is surrounded by a radially thin, perfectly conducting wall lying on the flux-surface $r = a$, where $a > b$. The region $b < r < a$ is a vacuum characterized by $g = 1$ and $P = 0$.

Let us launch J linearly independent, well-behaved solution vectors, $\mathbf{y}_j^e(r)$, for $j = 1, J$, from the magnetic axis, $r = 0$, and numerically integrate them to $r = a$. (It is generally necessary to periodically re-orthogonalize the solution vectors to prevent them from becoming co-linear. See Ref. 32, Appendix A.3.) The jump conditions imposed at the plasma rational surfaces are

$$\Psi_{k'}^o = 0, \quad (\text{A34})$$

$$\Delta \Psi_{k'}^e = 0, \quad (\text{A35})$$

for $k' = 1, K$. Suppose that there are L rational surfaces lying in the vacuum region $b < r < a$. The jump conditions imposed at the vacuum rational surfaces are

$$\Psi_{l'}^+ = \Psi_{l'}^-, \quad (\text{A36})$$

$$\Delta \Psi_{l'}^+ = \Delta \Psi_{l'}^-, \quad (\text{A37})$$

for $l' = 1, L$, where the $\Psi_{l'}^\pm$ and $\Delta \Psi_{l'}^\pm$ are defined in an analogous manner to the corresponding quantities for plasma rational surfaces (see Sect. A 5).

Next, let us launch a solution vector, $\Delta \mathbf{y}_k^e(r)$, from the k th plasma rational surface, and numerically integrate it to $r = a$. The jump conditions imposed at the plasma rational

surfaces are

$$\Psi_{k'}^o = 0, \quad (\text{A38})$$

$$\Delta\Psi_{k'}^e = \delta_{k'k}, \quad (\text{A39})$$

for $k' = 1, K$. The jump conditions (A36) and (A37) are again imposed at the vacuum rational surfaces.

We can form a linear combination of solution vectors,

$$\mathbf{Y}_k^e(r) = \sum_{j=1, J} \alpha_{jk}^e \mathbf{y}_j^e + \Delta\mathbf{y}_k^e, \quad (\text{A40})$$

and choose the α_{jk}^e so as to ensure that the physical boundary condition

$$\psi_j(a) = 0, \quad (\text{A41})$$

for $j = 1, J$, is satisfied. By construction, this solution vector is such that

$$\Psi_{k'}^o = 0. \quad (\text{A42})$$

$$\Delta\Psi_{k'}^e = \delta_{k'k}, \quad (\text{A43})$$

for $k' = 1, K$, and

$$\Psi_{l'}^+ = \Psi_{l'}^-, \quad (\text{A44})$$

$$\Delta\Psi_{l'}^+ = \Delta\Psi_{l'}^-, \quad (\text{A45})$$

for $l' = 1, L$. Incidentally, the jump conditions (A44) and (A45) ensure that the solution vector is completely continuous across the vacuum rational surfaces, as must be the case in the absence of plasma currents. (Consequently, zero electromagnetic torque is exerted at these surfaces.) Let

$$\Psi_{k'}^e = F_{k'k}^{ee}, \quad (\text{A46})$$

$$\Delta\Psi_{k'}^o = F_{k'k}^{oe}, \quad (\text{A47})$$

for $k' = 1, K$. We can associate a $\mathbf{Y}_k^e(r)$ with each rational surface in the plasma.

Let us launch J linearly independent, well-behaved solution vectors, $\mathbf{y}_j^o(r)$, for $j = 1, J$, from the magnetic axis, $r = 0$, and numerically integrate them to $r = a$. The jump conditions

imposed at the plasma rational surfaces are

$$\Psi_{k'}^e = 0, \quad (\text{A48})$$

$$\Delta\Psi_{k'}^o = 0, \quad (\text{A49})$$

for $k' = 1, K$. The jump conditions (A36) and (A37) are again imposed at the vacuum rational surfaces.

Next, we can launch a solution vector, $\Delta\mathbf{y}_k^o(r)$, from the k th rational surface, and integrate it to $r = a$. The jump conditions imposed at the plasma rational surfaces are

$$\Psi_{k'}^e = 0, \quad (\text{A50})$$

$$\Delta\Psi_{k'}^o = \delta_{k'k}, \quad (\text{A51})$$

for $k = 1, K$. As before, the jump conditions (A36) and (A37) are imposed at the vacuum rational surfaces.

We can form the linear combination of solution vectors,

$$\mathbf{Y}_k^o(r) = \sum_{j=1, J} \alpha_{jk}^o \mathbf{y}_j^o + \Delta\mathbf{y}_k^o, \quad (\text{A52})$$

and choose the α_{jk}^o so as to satisfy the boundary condition (A41). By construction, this solution vector is such that

$$\Psi_{k'}^e = 0, \quad (\text{A53})$$

$$\Delta\Psi_{k'}^o = \delta_{k'k}, \quad (\text{A54})$$

for $k' = 1, K$, and

$$\Psi_{l'}^+ = \Psi_{l'}^-, \quad (\text{A55})$$

$$\Delta\Psi_{l'}^+ = \Psi_{l'}^-, \quad (\text{A56})$$

for $l' = 1, L$. Let

$$\Psi_{k'}^o = F_{k'k}^{oo}, \quad (\text{A57})$$

$$\Delta\Psi_{k'}^e = F_{k'k}^{eo}, \quad (\text{A58})$$

for $k' = 1, K$. We can associate a $\mathbf{Y}_k^o(r)$ with each rational surface in the plasma.

The most general well-behaved solution vector that satisfies the boundary condition (A41) is written

$$\mathbf{Y}(r) = \sum_{k=1,K} (a_k \mathbf{Y}_k^e + b_k \mathbf{Y}_k^o), \quad (\text{A59})$$

where the a_k and b_k are arbitrary. It follows that

$$\Psi_k^e = \sum_{k'=1,K} F_{kk'}^{ee} a_{k'}, \quad (\text{A60})$$

$$\Psi_k^o = \sum_{k'=1,K} F_{kk'}^{oo} b_{k'}, \quad (\text{A61})$$

$$\Delta\Psi_k^e = a_k + \sum_{k'=1,K} F_{kk'}^{eo} b_{k'}, \quad (\text{A62})$$

$$\Delta\Psi_k^o = b_k + \sum_{k'=1,K} F_{kk'}^{oe} a_{k'}, \quad (\text{A63})$$

for $k = 1, K$. Let Ψ^e , Ψ^o , $\Delta\Psi^e$, and $\Delta\Psi^o$ be the $K \times 1$ vectors of the Ψ_k^e , Ψ_k^o , $\Delta\Psi_k^e$, and $\Delta\Psi_k^o$ values, respectively. Let \mathbf{F}^{ee} , \mathbf{F}^{eo} , \mathbf{F}^{oe} , and \mathbf{F}^{oo} be the $K \times K$ matrices of the $F_{kk'}^{ee}$, $F_{kk'}^{eo}$, $F_{kk'}^{oe}$, and $F_{kk'}^{oo}$ values, respectively. Equations (A60)–(A63) can be combined to give the dispersion relation

$$\begin{pmatrix} \Delta\Psi^e \\ \Delta\Psi^o \end{pmatrix} = \begin{pmatrix} \mathbf{E}^e & \mathbf{\Gamma} \\ \mathbf{\Gamma}' & \mathbf{E}^o \end{pmatrix} \begin{pmatrix} \Psi^e \\ \Psi^o \end{pmatrix}, \quad (\text{A64})$$

where

$$\mathbf{E}^e = (\mathbf{F}^{ee})^{-1}, \quad (\text{A65})$$

$$\mathbf{E}^o = (\mathbf{F}^{oo})^{-1}, \quad (\text{A66})$$

$$\mathbf{\Gamma} = \mathbf{F}^{eo} \mathbf{E}^o, \quad (\text{A67})$$

$$\mathbf{\Gamma}' = \mathbf{F}^{oe} \mathbf{E}^e. \quad (\text{A68})$$

Now, according to Eqs. (A21) and (A41),

$$T_\phi(a) = \sum_{k=1,K} \delta T_k = 0. \quad (\text{A69})$$

In other words, the net toroidal electromagnetic torque acting on the plasma is zero. Hence, Eq. (A33) yields

$$\Psi^{e\dagger} \Delta\Psi^e - \Delta\Psi^{e\dagger} \Psi^e + \Psi^{o\dagger} \Delta\Psi^o - \Delta\Psi^{o\dagger} \Psi^o = 0. \quad (\text{A70})$$

Thus, making use of the dispersion relation (A64), we deduce that^{19–21,27}

$$\mathbf{E}^{e\dagger} = \mathbf{E}^e, \quad (\text{A71})$$

$$\mathbf{E}^{o\dagger} = \mathbf{E}^o, \quad (\text{A72})$$

$$\mathbf{\Gamma}' = \mathbf{\Gamma}^\dagger. \quad (\text{A73})$$

Thus, the homogeneous dispersion relation can be written^{19,21,26,27}

$$\begin{pmatrix} \mathbf{E}^e - \mathbf{\Delta}^e & \mathbf{\Gamma} \\ \mathbf{\Gamma}' & \mathbf{E}^o - \mathbf{\Delta}^o \end{pmatrix} \begin{pmatrix} \mathbf{\Psi}^e \\ \mathbf{\Psi}^o \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \quad (\text{A74})$$

where $\mathbf{\Delta}^e$ and $\mathbf{\Delta}^o$ are the diagonal $K \times K$ matrices of the Δ_k^e and Δ_k^o values, respectively. Note that, according to Eqs. (A71)–(A73), the \mathbf{E}^e and \mathbf{E}^o matrices are Hermitian, and the $\mathbf{\Gamma}'$ matrix is the Hermitian conjugate of the $\mathbf{\Gamma}$ matrix.

7. Derivation of Inhomogeneous Dispersion Relation

Let us define the solution vectors

$$\tilde{\mathbf{Y}}_k^e(r) = \sum_{k''=1,K} E_{k''k}^e \mathbf{Y}_{k''}^e, \quad (\text{A75})$$

for $k = 1, K$. It follows, from Sect. A 6, that these vectors are well-behaved solutions of the outer equations, satisfying the physical boundary condition at the wall, and having the properties that

$$\Psi_{k'}^e = \delta_{k'k}, \quad (\text{A76})$$

$$\Psi_{k'}^o = 0, \quad (\text{A77})$$

$$\Delta \Psi_{k'}^e = E_{k'k}^e, \quad (\text{A78})$$

$$\Delta \Psi_{k'}^o = \Gamma_{kk'}^*, \quad (\text{A79})$$

for $k' = 1, K$. Let the $\tilde{\psi}_{k,j}^e(r)$ and the $\tilde{Z}_{k,j}^e(r)$ be the elements of the $\tilde{\mathbf{Y}}_k^e(r)$ solution vector, for $j = 1, J$. By construction, we have

$$\tilde{\psi}_{k,j}^e(a) = 0, \quad (\text{A80})$$

for $j = 1, J$.

Let us define the solution vectors

$$\tilde{\mathbf{Y}}_k^o(r) = \sum_{k''=1,K} E_{k''k}^o \mathbf{Y}_{k''}^o, \quad (\text{A81})$$

for $k = 1, K$. It follows, from Sect. A 6, that these vectors are well-behaved solutions of the outer equations, satisfying the physical boundary condition at the wall, and having the properties that

$$\Psi_{k'}^e = 0, \quad (\text{A82})$$

$$\Psi_{k'}^o = \delta_{k'k}, \quad (\text{A83})$$

$$\Delta \Psi_{k'}^e = \Gamma_{k'k}, \quad (\text{A84})$$

$$\Delta \Psi_{k'}^o = E_{k'k}^o, \quad (\text{A85})$$

for $k' = 1, K$. Let the $\tilde{\psi}_{k,j}^o(r)$ and the $\tilde{Z}_{k,j}^o(r)$ be the elements of the $\tilde{\mathbf{Y}}_k^o(r)$ solution vector, for $j = 1, J$. By construction, we have

$$\tilde{\psi}_{k,j}^o(a) = 0, \quad (\text{A86})$$

for $j = 1, J$.

The most general well-behaved solution vector in the presence of an external magnetic perturbation is written

$$\mathbf{Y}(r) = \sum_{k=1,K} \left(\Psi_k^e \tilde{\mathbf{Y}}_k^e + \Psi_k^o \tilde{\mathbf{Y}}_k^o \right) + \mathbf{Y}^x, \quad (\text{A87})$$

where $\mathbf{Y}^x(r)$ is the solution vector that describes the ideal (i.e., $\Psi_k^e = \Psi_k^o = 0$ for $k = 1, K$) response of the plasma to the perturbation. This solution vector is characterized by

$$\Psi_{k'}^e = 0, \quad (\text{A88})$$

$$\Psi_{k'}^o = 0, \quad (\text{A89})$$

$$\Delta \Psi_{k'}^e = -\chi_{k'}^e, \quad (\text{A90})$$

$$\Delta \Psi_{k'}^o = -\chi_{k'}^o, \quad (\text{A91})$$

for $k' = 0, K$. Thus, in the presence of the external perturbation, the homogenous dispersion relation (A64) generalizes to the inhomogenous relation

$$\begin{pmatrix} \mathbf{E}^e - \Delta^e & \Gamma \\ \Gamma' & \mathbf{E}^o - \Delta^o \end{pmatrix} \begin{pmatrix} \Psi^e \\ \Psi^o \end{pmatrix} = \begin{pmatrix} \chi^e \\ \chi^o \end{pmatrix}, \quad (\text{A92})$$

where $\boldsymbol{\chi}^e$ and $\boldsymbol{\chi}^o$ are the $K \times 1$ vectors of the χ_k^e and χ_k^o values, respectively.

Let the $\psi_j^x(r)$ and the $Z_j^x(r)$ be the elements of the $\mathbf{Y}^x(r)$ solution vector, for $j = 1, J$. Suppose that

$$\psi_j^x(a) = \frac{a}{q(a)} [m_j - n q(a)] \Xi_j, \quad (\text{A93})$$

$$Z_j^x(a) = \Omega_j, \quad (\text{A94})$$

for $j = 1, J$. According to Eq. (A21), the net toroidal electromagnetic torque acting on the plasma is written²³

$$T_\phi(a) = \sum_{k=1, K} \delta T_k = -2 n \pi^2 \text{Im} \sum_{j=1, J} \frac{Z_j^* \psi_j}{m_j - n q} \Big|_{r=a}. \quad (\text{A95})$$

However,

$$\psi_j(a) = \frac{a}{q(a)} [m_j - n q(a)] \Xi_j, \quad (\text{A96})$$

$$Z_j(a) = \sum_{k=1, K} \left[\Psi_k^e \tilde{Z}_{k,j}^e(a) + \Psi_k^o \tilde{Z}_{k,j}^o(a) \right] + \Omega_j, \quad (\text{A97})$$

so

$$T_\phi(a) = -2 n \pi^2 \text{Im} \sum_{J=1, J} \sum_{k=1, K} \frac{a}{q(a)} \Xi_j \left[\Psi_k^e \tilde{Z}_{k,j}^e(a) + \Psi_k^o \tilde{Z}_{k,j}^o(a) \right]^*. \quad (\text{A98})$$

Here, we have assumed that

$$\text{Im} \sum_{j=1, J} \Omega_j^* \Xi_j = 0, \quad (\text{A99})$$

otherwise the external perturbation would be able to exert a torque on an ideal plasma (i.e., a plasma in which $\Psi_k^e = \Psi_k^o = 0$, for $k = 1, K$), which is unphysical.

It follows from Eqs. (A33) and (A92), as well as the fact that \mathbf{E}^e and \mathbf{E}^o matrices are Hermitian, that

$$T_\phi(a) = \sum_{k=1, K} \delta T_k = -2 n \pi^2 \text{Im} \sum_{k=1, K} (\Psi_k^{e*} \chi_k^e + \Psi_k^{o*} \chi_k^o). \quad (\text{A100})$$

Comparison with Eq. (A98) reveals that

$$\chi_k^e = \sum_{j=1, J} \Xi_j \xi_{k,j}^{e*}, \quad (\text{A101})$$

$$\chi_k^o = \sum_{j=1, J} \Xi_j \xi_{k,j}^{o*}, \quad (\text{A102})$$

where

$$\xi_{k,j}^e = \frac{a}{q(a)} \tilde{Z}_{k,j}^e(a), \quad (\text{A103})$$

$$\xi_{k,j}^o = \frac{a}{q(a)} \tilde{Z}_{k,j}^o(a). \quad (\text{A104})$$

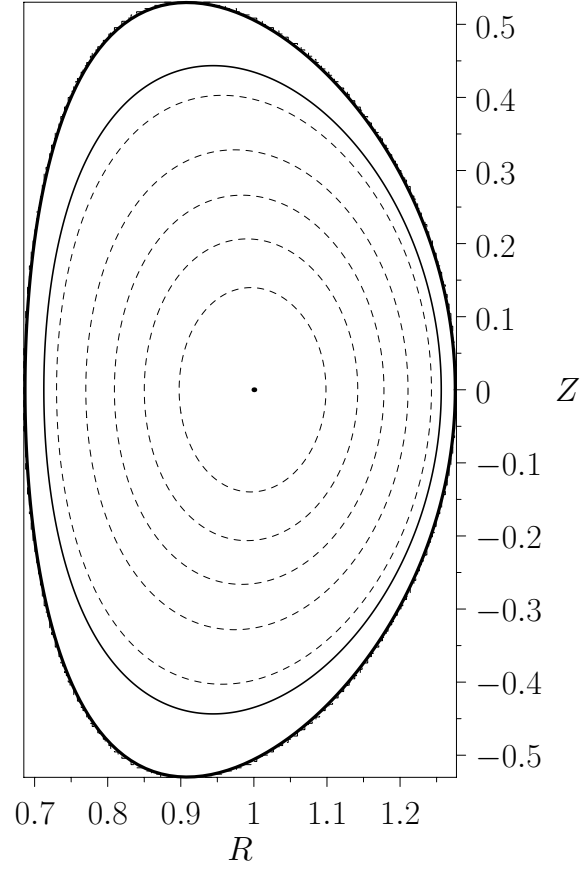


FIG. 1. Unperturbed magnetic flux-surfaces for a plasma equilibrium characterized by $\bar{a} = 0.3$, $\kappa = 1.8$, $\bar{\delta} = 0.25$, $s_b = 0.95$, $q_0 = 1.05$, $q_b = 3.95$, and $\beta_N = 0.25$. The thick solid curve shows the location of the perfectly conducting wall surrounding the plasma, the thin solid curve shows the location of the plasma boundary, and the dashed curves show magnetic flux-surfaces inside the plasma.

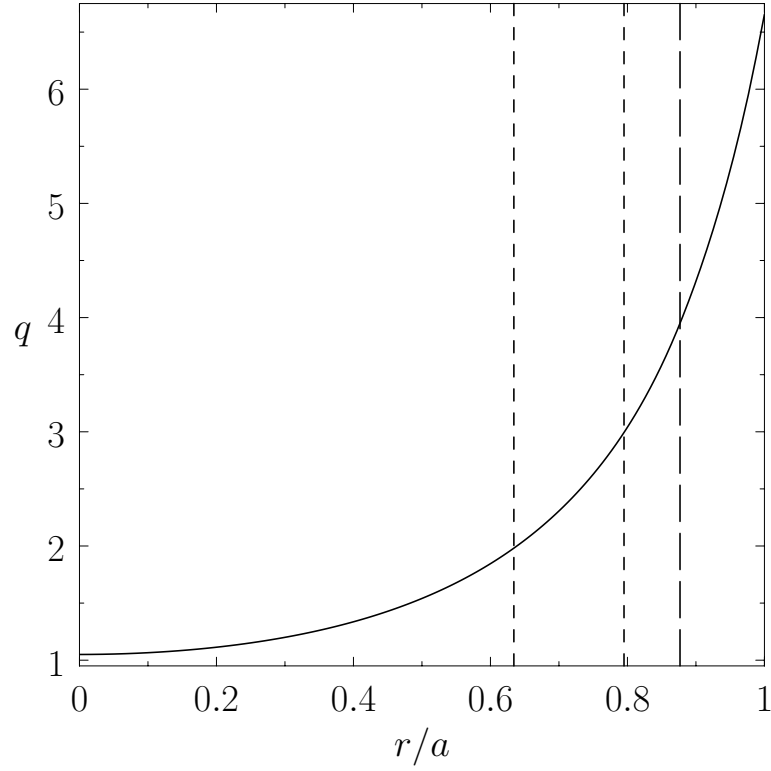


FIG. 2. Safety-factor profile for a plasma equilibrium characterized by $\bar{a} = 0.3$, $\kappa = 1.8$, $\bar{\delta} = 0.25$, $s_b = 0.95$, $q_0 = 1.05$, $q_b = 3.95$, and $\beta_N = 0.25$. The long-dashed vertical line indicates the location of the plasma boundary. The two short-dashed vertical lines indicate the locations of the $q = 2$ and $q = 3$ surfaces.

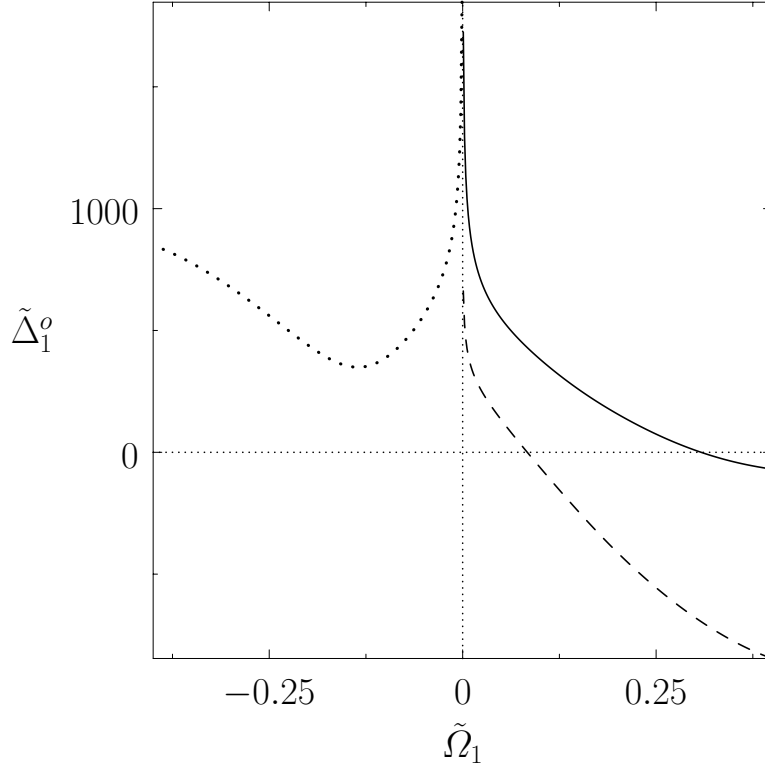


FIG. 3. Variation of the $n = 1$ twisting-parity layer response function at the $q = 2$ surface, $\tilde{\Delta}_1^o$, with the normalized local plasma toroidal angular velocity, $\tilde{\Omega}_1$, for a plasma equilibrium characterized by $\bar{a} = 0.3$, $\kappa = 1.8$, $\bar{\delta} = 0.25$, $s_b = 0.95$, $q_0 = 1.05$, $q_b = 3.95$, and $\beta_N = 0.25$. The solid and dashed curves on the right-hand side show $\text{Re}(\tilde{\Delta}_1^o)$ and $\text{Im}(\tilde{\Delta}_1^o)$, respectively. The dotted curve on the left-hand side shows $|\tilde{\Delta}_1^o|$. Note that $\text{Re}(\tilde{\Delta}_1^o)$ and $\text{Im}(\tilde{\Delta}_1^o)$ are even and odd functions of $\tilde{\Omega}_1$, respectively.

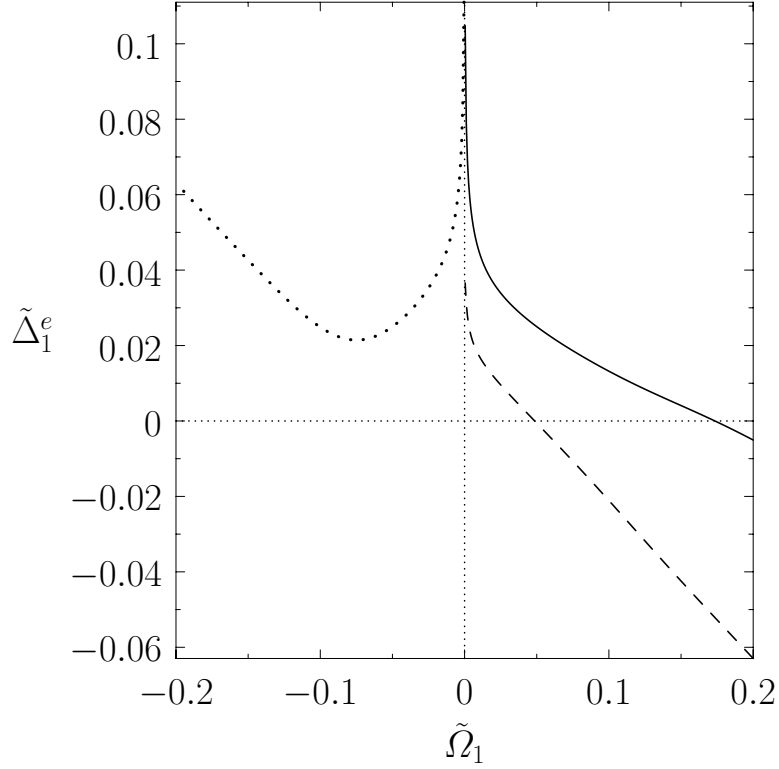


FIG. 4. Variation of the $n = 1$ tearing-parity layer response function at the $q = 2$ surface, $\tilde{\Delta}_1^e$, with the normalized local plasma toroidal angular velocity, $\tilde{\Omega}_1$, for a plasma equilibrium characterized by $\bar{a} = 0.3$, $\kappa = 1.8$, $\bar{\delta} = 0.25$, $s_b = 0.95$, $q_0 = 1.05$, $q_b = 3.95$, and $\beta_N = 0.25$. The solid and dashed curves on the right-hand side show $\text{Re}(\tilde{\Delta}_1^e)$ and $\text{Im}(\tilde{\Delta}_1^e)$, respectively. The dotted curve on the left-hand side shows $|\tilde{\Delta}_1^e|$. Note that $\text{Re}(\tilde{\Delta}_1^e)$ and $\text{Im}(\tilde{\Delta}_1^e)$ are even and odd functions of $\tilde{\Omega}_1$, respectively.

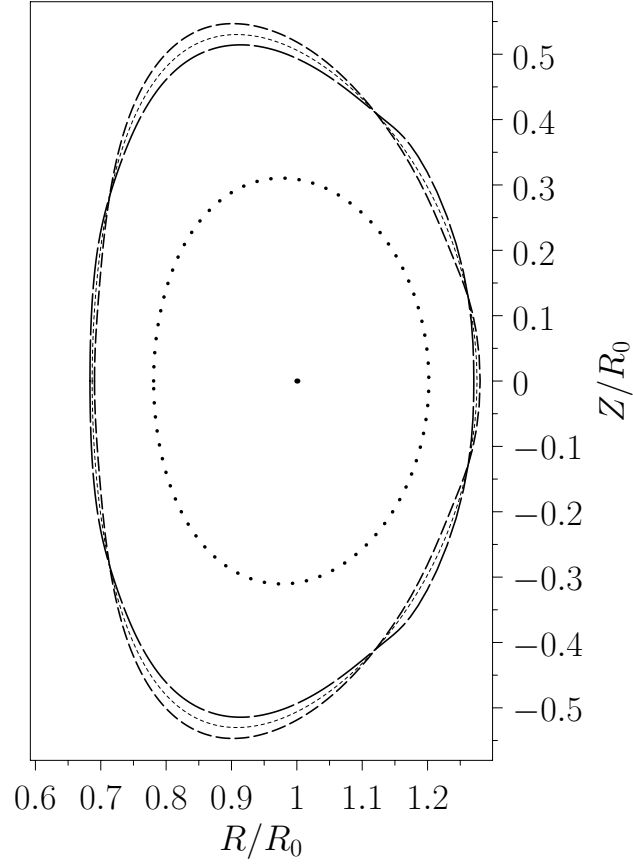


FIG. 5. The optimal $n = 1$ wall displacement for driving tearing-parity magnetic flux at the $q = 2$ surface calculated for a plasma equilibrium characterized by $\bar{a} = 0.3$, $\kappa = 1.8$, $\bar{\delta} = 0.25$, $s_b = 0.95$, $q_0 = 1.05$, $q_b = 3.95$, and $\beta_N = 0.25$. The thin dashed curve shows $r = a$. The thick dashed curves show $r = a \pm \delta C_1(\theta)$, where $\delta = 0.025$. Finally, the dotted curve shows the location of the $q = 2$ surface.

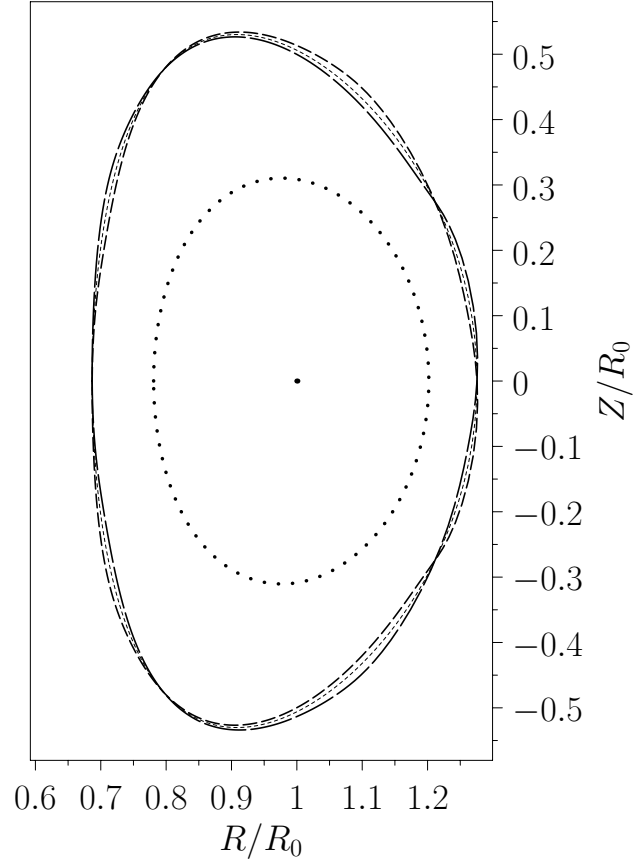


FIG. 6. The optimal $n = 1$ wall displacement for driving tearing-parity magnetic flux at the $q = 2$ surface calculated for a plasma equilibrium characterized by $\bar{a} = 0.3$, $\kappa = 1.8$, $\bar{\delta} = 0.25$, $s_b = 0.95$, $q_0 = 1.05$, $q_b = 3.95$, and $\beta_N = 0.25$. The thin dashed curve shows $r = a$. The thick dashed curves show $r = a \pm \delta S_1(\theta)$, where $\delta = 0.025$. Finally, the dotted curve shows the location of the $q = 2$ surface.

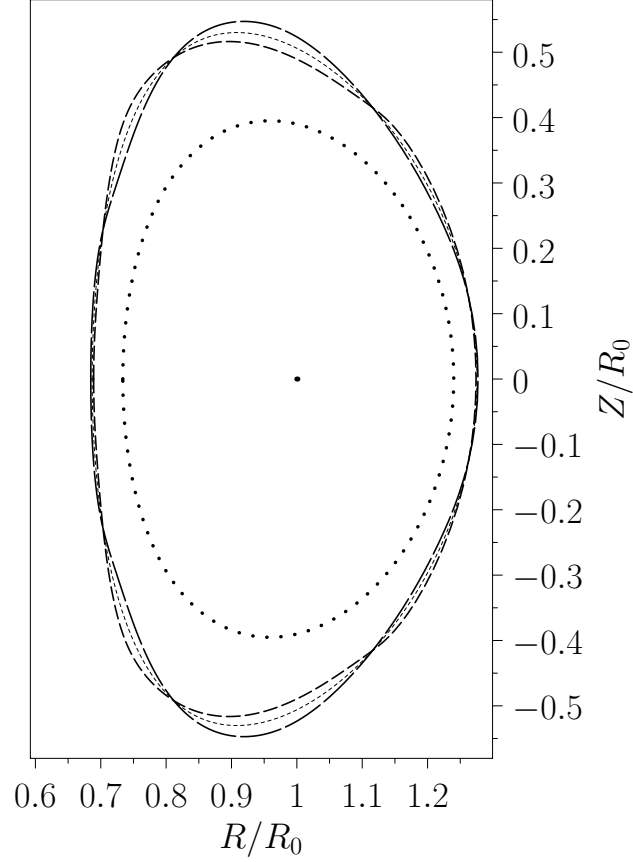


FIG. 7. The optimal $n = 1$ wall displacement for driving tearing-parity magnetic flux at the $q = 3$ surface calculated for a plasma equilibrium characterized by $\bar{a} = 0.3$, $\kappa = 1.8$, $\bar{\delta} = 0.25$, $s_b = 0.95$, $q_0 = 1.05$, $q_b = 3.95$, and $\beta_N = 0.25$. The thin dashed curve shows $r = a$. The thick dashed curves show $r = a \pm \delta C_2(\theta)$, where $\delta = 0.025$. Finally, the dotted curve shows the location of the $q = 3$ surface.

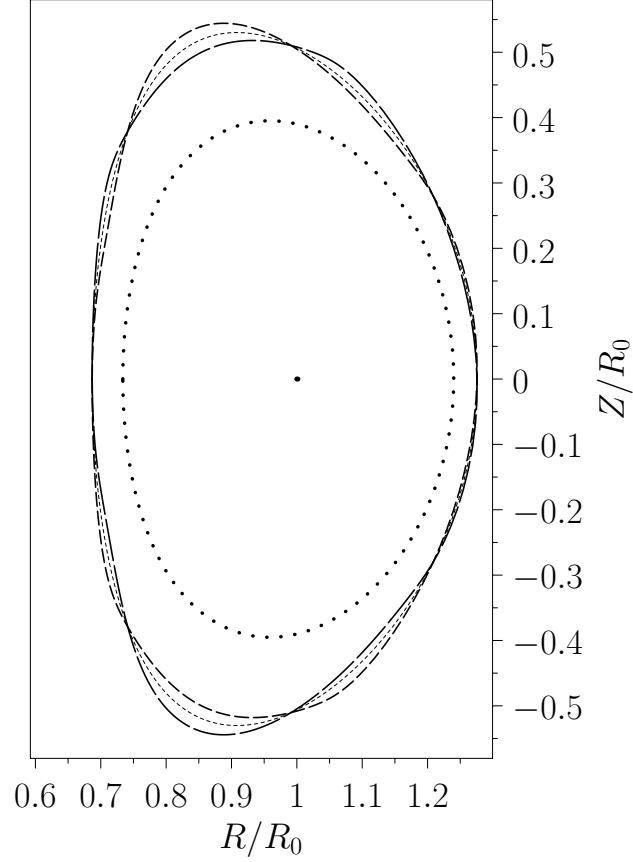


FIG. 8. The optimal $n = 1$ wall displacement for driving tearing-parity magnetic flux at the $q = 3$ surface calculated for a plasma equilibrium characterized by $\bar{a} = 0.3$, $\kappa = 1.8$, $\bar{\delta} = 0.25$, $s_b = 0.95$, $q_0 = 1.05$, $q_b = 3.95$, and $\beta_N = 0.25$. The thin dashed curve shows $r = a$. The thick dashed curves show $r = a \pm \delta S_2(\theta)$, where $\delta = 0.025$. Finally, the dotted curve shows the location of the $q = 3$ surface.

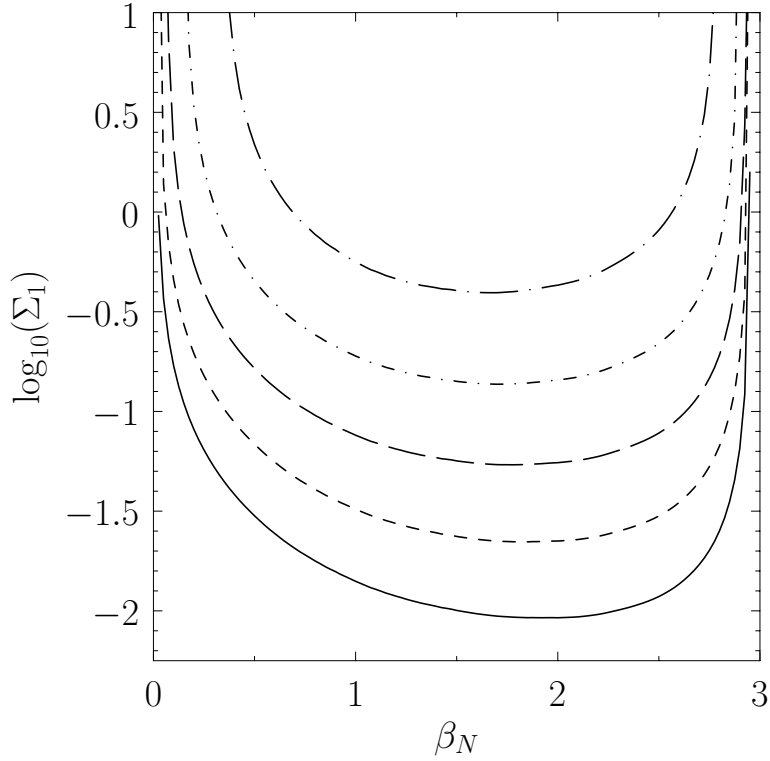


FIG. 9. The shielding factor for tearing-parity magnetic flux driven at the $q = 2$ surface by an $n = 1$ wall displacement calculated as a function of the normal beta for a series of plasma equilibria characterized by $\bar{a} = 0.3$, $\kappa = 1.8$, $\bar{\delta} = 0.25$, $s_b = 0.95$, $q_0 = 1.05$, and $q_b = 3.95$. The solid, short-dashed, long-dashed, dot-short-dashed, and dot-long-dashed curves correspond to $S_1 = 10^{10}$, 10^9 , 10^8 , 10^7 , and 10^6 , respectively.

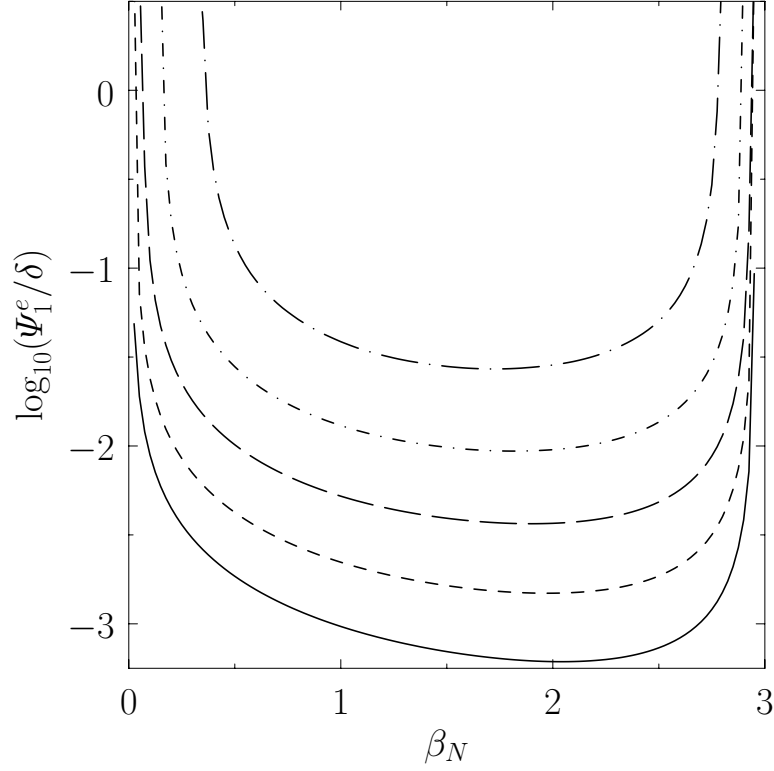


FIG. 10. The ratio of the tearing-parity magnetic flux driven at the $q = 2$ surface by an optimal $n = 1$ wall displacement to the mean wall displacement calculated as a function of the normal beta for a series of plasma equilibria characterized by $\bar{a} = 0.3$, $\kappa = 1.8$, $\bar{\delta} = 0.25$, $s_b = 0.95$, $q_0 = 1.05$, and $q_b = 3.95$. The solid, short-dashed, long-dashed, dot-short-dashed, and dot-long-dashed curves correspond to $S_1 = 10^{10}$, 10^9 , 10^8 , 10^7 , and 10^6 , respectively.

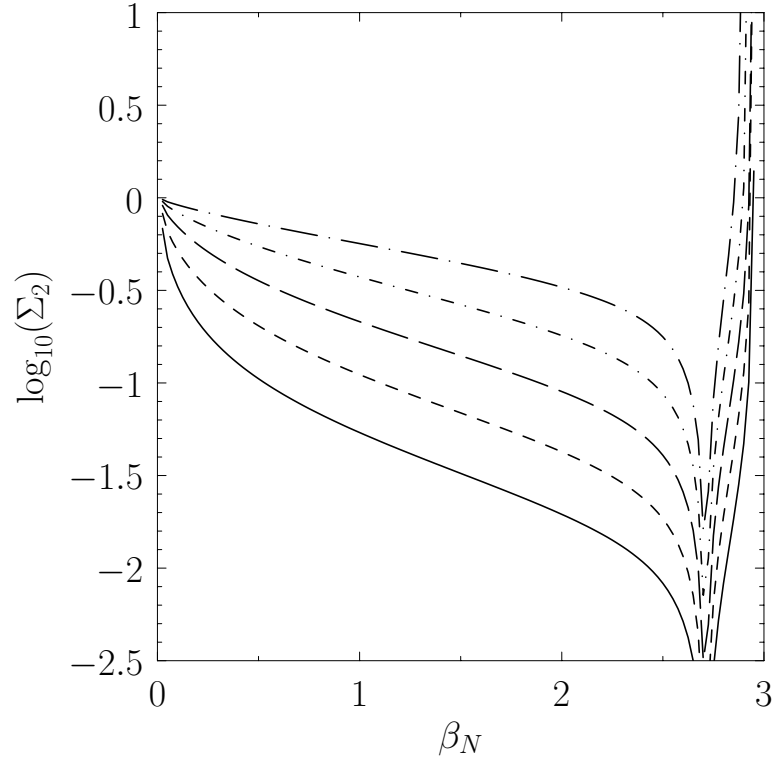


FIG. 11. The shielding factor for tearing-parity magnetic flux driven at the $q = 3$ surface by an $n = 1$ wall displacement calculated as a function of the normal beta for a series of plasma equilibria characterized by $\bar{a} = 0.3$, $\kappa = 1.8$, $\bar{\delta} = 0.25$, $s_b = 0.95$, $q_0 = 1.05$, and $q_b = 3.95$. The solid, short-dashed, long-dashed, dot-short-dashed, and dot-long-dashed curves correspond to $S_1 = 10^{10}$, 10^9 , 10^8 , 10^7 , and 10^6 , respectively.

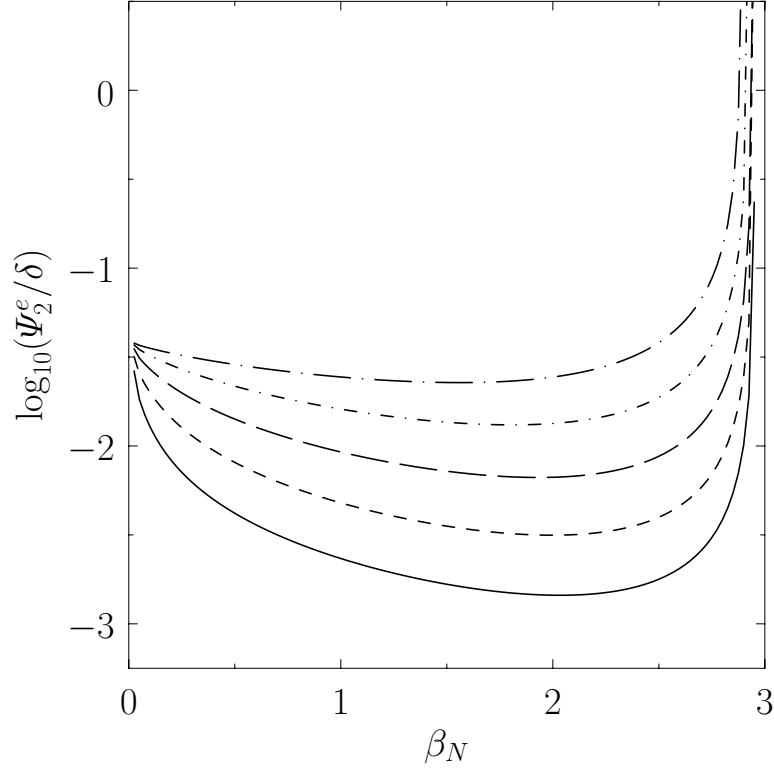


FIG. 12. The ratio of the tearing-parity magnetic flux driven at the $q = 3$ surface by an optimal $n = 1$ wall displacement to the mean wall displacement calculated as a function of the normal beta for a series of plasma equilibria characterized by $\bar{a} = 0.3$, $\kappa = 1.8$, $\bar{\delta} = 0.25$, $s_b = 0.95$, $q_0 = 1.05$, and $q_b = 3.95$. The solid, short-dashed, long-dashed, dot-short-dashed, and dot-long-dashed curves correspond to $S_1 = 10^{10}$, 10^9 , 10^8 , 10^7 , and 10^6 , respectively.

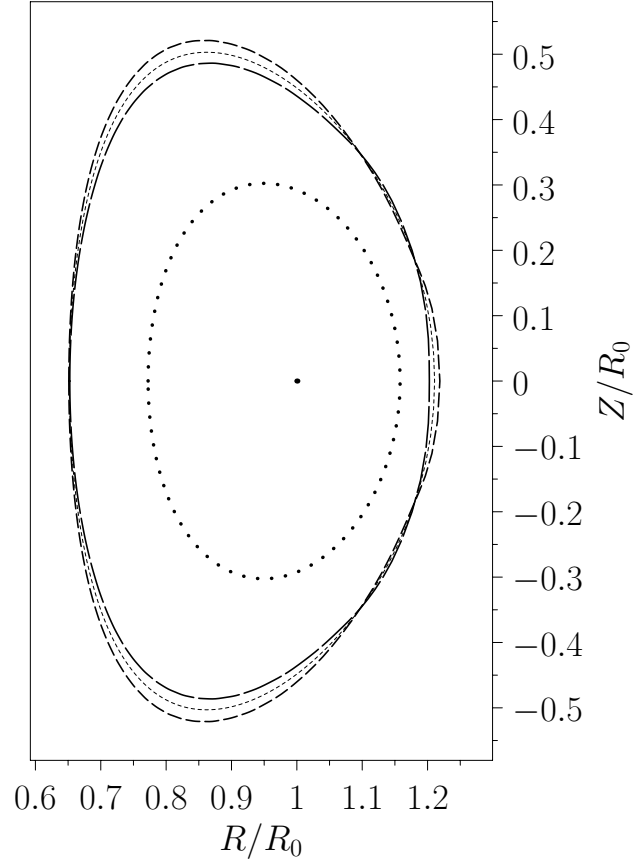


FIG. 13. The optimal $n = 1$ wall displacement for driving tearing-parity magnetic flux at the $q = 2$ surface calculated for a plasma equilibrium characterized by $\bar{a} = 0.3$, $\kappa = 1.8$, $\bar{\delta} = 0.25$, $s_b = 0.95$, $q_0 = 1.05$, $q_b = 3.95$, and $\beta_N = 3.95$. The thin dashed curve shows $r = a$. The thick dashed curves show $r = a \pm \delta C_1(\theta)$, where $\delta = 0.025$. Finally, the dotted curve shows the location of the $q = 2$ surface.

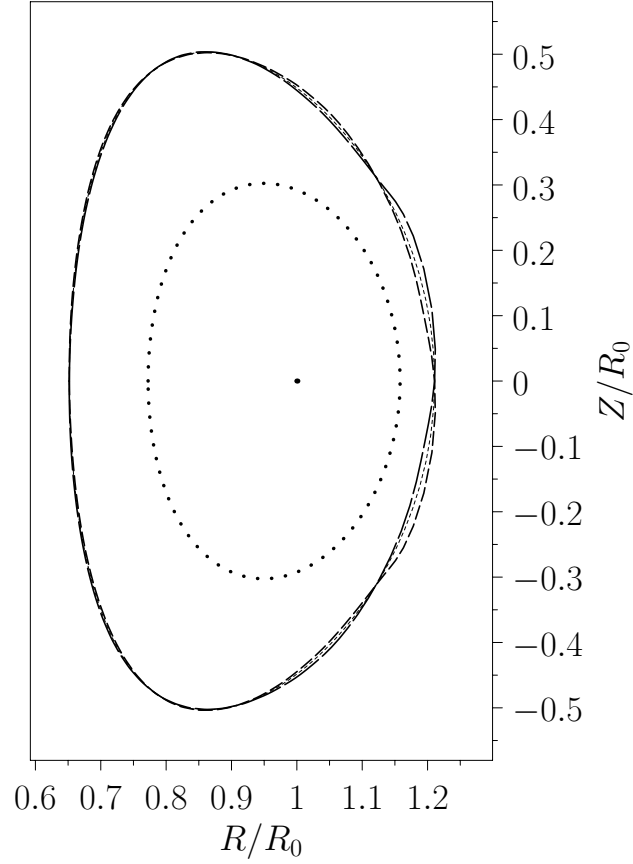


FIG. 14. The optimal $n = 1$ wall displacement for driving tearing-parity magnetic flux at the $q = 2$ surface calculated for a plasma equilibrium characterized by $\bar{a} = 0.3$, $\kappa = 1.8$, $\bar{\delta} = 0.25$, $s_b = 0.95$, $q_0 = 1.05$, $q_b = 3.95$, and $\beta_N = 3.95$. The thin dashed curve shows $r = a$. The thick dashed curves show $r = a \pm \delta S_1(\theta)$, where $\delta = 0.025$. Finally, the dotted curve shows the location of the $q = 2$ surface.

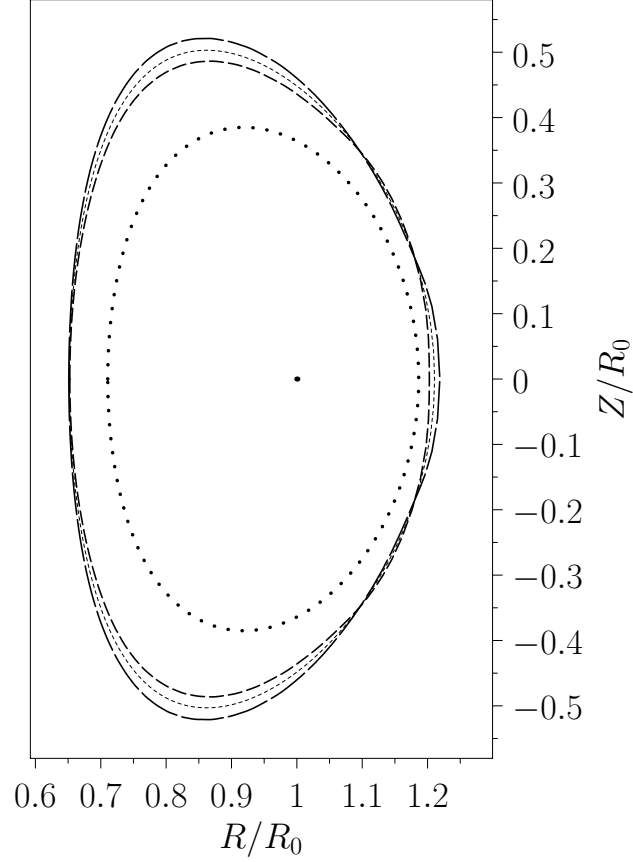


FIG. 15. The optimal $n = 1$ wall displacement for driving tearing-parity magnetic flux at the $q = 3$ surface calculated for a plasma equilibrium characterized by $\bar{a} = 0.3$, $\kappa = 1.8$, $\bar{\delta} = 0.25$, $s_b = 0.95$, $q_0 = 1.05$, $q_b = 3.95$, and $\beta_N = 3.95$. The thin dashed curve shows $r = a$. The thick dashed curves show $r = a \pm \delta C_2(\theta)$, where $\delta = 0.025$. Finally, the dotted curve shows the location of the $q = 3$ surface.

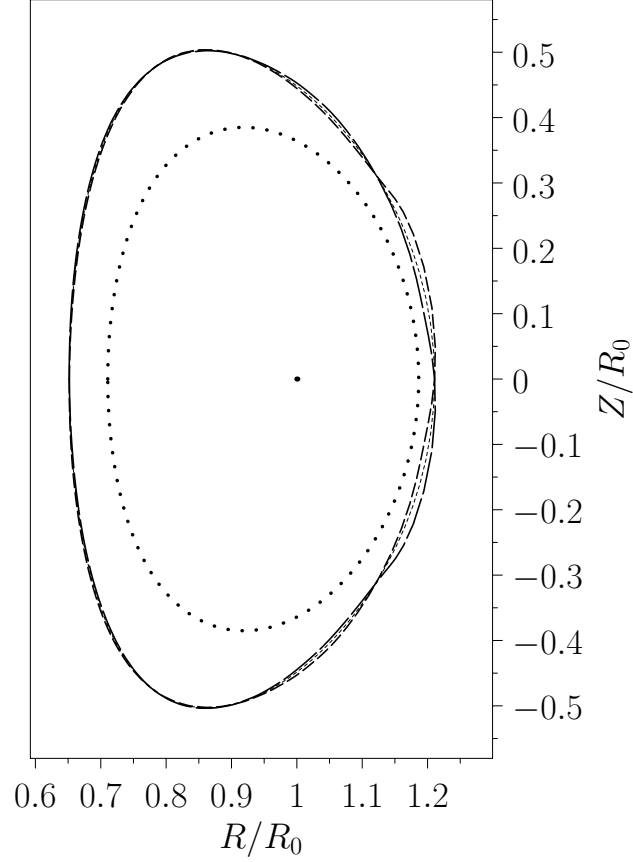


FIG. 16. The optimal $n = 1$ wall displacement for driving tearing-parity magnetic flux at the $q = 3$ surface calculated for a plasma equilibrium characterized by $\bar{a} = 0.3$, $\kappa = 1.8$, $\bar{\delta} = 0.25$, $s_b = 0.95$, $q_0 = 1.05$, $q_b = 3.95$, and $\beta_N = 3.95$. The thin dashed curve shows $r = a$. The thick dashed curves show $r = a \pm \delta S_2(\theta)$, where $\delta = 0.025$. Finally, the dotted curve shows the location of the $q = 3$ surface.

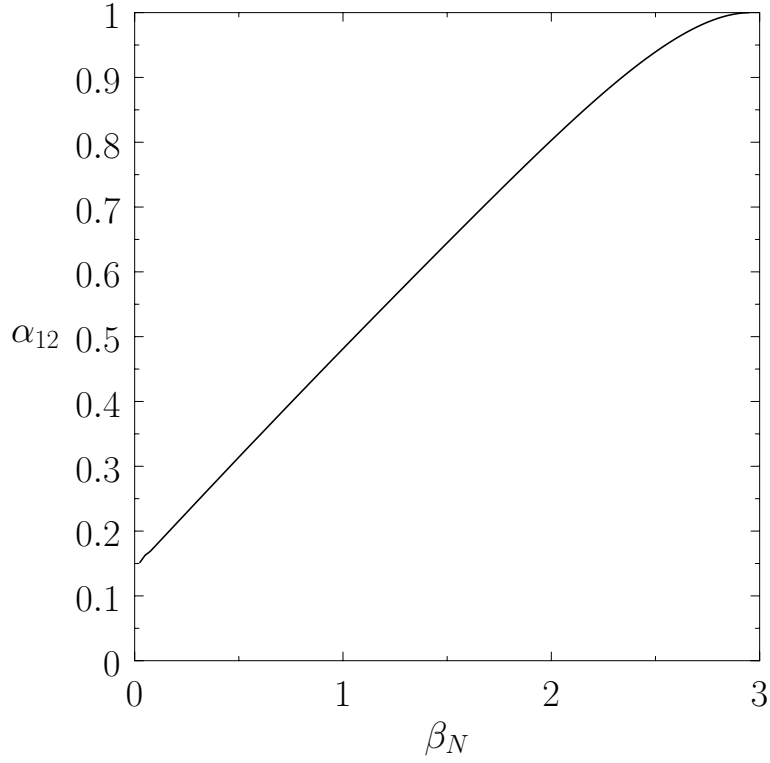


FIG. 17. The correlation factor between the optimal $n = 1$ wall displacement shapes required to drive tearing-parity magnetic flux at the $q = 2$ and the $q = 3$ surfaces calculated as a function of the normal beta for a series of plasma equilibria characterized by $\bar{a} = 0.3$, $\kappa = 1.8$, $\bar{\delta} = 0.25$, $s_b = 0.95$, $q_0 = 1.05$, and $q_b = 3.95$.