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Collisionless magnetic reconnection with arbitrary guide field

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A set of reduced equations governing nonlinear, two-dimensional, two-fluid, collisionless magnetic reconnection with arbitrary guide field is derived. These equations represent an improvement on the existing nonlinear reduced equations generally used to investigate collisionless reconnection, which are only valid in the large guide-field limit. The improved equations are used to calculate the linear growth rate of a strongly unstable, spontaneously reconnecting, plasma instability, as well as the general linear dispersion relation for such an instability. © *2004 American Institute of Physics*. [DOI: 10.1063/1.1791640]

I. INTRODUCTION

Magnetic reconnection is a fundamental physical phenomenon which occurs in a wide variety of laboratory and space plasmas, e.g., magnetic fusion experiments,[1] the solar corona,[2] and the Earth's magnetotail.[3] The reconnection process gives rise to a change in magnetic field-line topology with an accompanying release of magnetic energy. Conventional collisional magnetohydrodynamical (MHD) theory is capable of accounting for magnetic reconnection, but generally predicts reconnection rates which are many orders of magnitude smaller than those observed in high-temperature plasmas.[4] On the other hand, more sophisticated plasma models that neglect collisions (which are irrelevant in high-temperature plasmas) and treat electrons and ions as separate fluids yield much faster reconnection rates which are fairly consistent with observations.[5,6]

Magnetic reconnection is generally thought of as a basically two-dimensional (2D) phenomenon in which oppositely directed magnetic field lines spontaneously merge together. However, in many cases of interest, there is also an essentially uniform magnetic field directed *perpendicular* to the merging field lines. This latter field is usually termed the "guide field." Unfortunately, the addition of a guide field significantly complicates the equations governing 2D, two-fluid, collisionless magnetic reconnection. In particular, the guide field couples the compressional Alfvén (CA) wave into these equations, despite the fact that this wave does not play a particularly significant role in magnetic reconnection. Furthermore, the CA wave becomes increasingly problematic (especially in numerical simulations) as the strength of the guide field increases. Now, it is a common practice in conventional MHD theory to "reduce" the equations governing MHD phenomena by removing the CA wave.[7] Indeed, this approach has been very successful over the years. In this paper, we present a set of reduced equations governing nonlinear, 2D, two-fluid, collisionless magnetic reconnection which is valid for both large and small guide fields (measured with respect to the reconnecting field). Our equations represent an improvement on the existing nonlinear reduced equations generally used to investigate collisionless reconnection, which are only valid in the large guide-field limit.[8,9] In addition, we calculate the linear growth rate of a strongly unstable, spontaneously reconnecting, plasma instability from our equations, as well as the general linear dispersion relation for such an instability.

II. DERIVATION OF REDUCED EQUATIONS

A. Basic equations

Standard right-handed Cartesian coordinates (x,y,z) are adopted. In the following, we employ a normalization scheme such that all lengths are normalized to a convenient length scale a, all magnetic field to a convenient scale field strength B_0, and all times to $\tau_A = a/V_A$, where $V_A = B_0/\sqrt{\mu_0 n m_i}$. Here, n is the electron/ion number density, which is assumed to be constant. We also assume that the ions are singly charged. Furthermore, m_i is the ion mass. Finally, the electrons are assumed to be hot and the ions cold.

The normalized electron equation of motion takes the form[10]

$$\epsilon d_i \left[\frac{\partial \boldsymbol{V}_e}{\partial t} + (\boldsymbol{V}_e \cdot \nabla) \boldsymbol{V}_e \right] = -d_i \nabla P - (\boldsymbol{E} + \boldsymbol{V}_e \times \boldsymbol{B}), \qquad (1)$$

where $\boldsymbol{E}$ is the electric field, $\boldsymbol{B}$ the magnetic field, $\boldsymbol{V}_e$ the electron velocity, P the electron pressure (which is assumed to be scalar), $d_i = (c/\omega_{pi})/a$ (where $\omega_{pi} = \sqrt{ne^2/\epsilon_0 m_i}$) the normalized collisionless ion skin depth, and $\epsilon = m_e/m_i$ the electron-ion mass ratio.

The normalized sum of the electron and ion equations of motion yields (neglecting electron inertia)

$$\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \nabla) \boldsymbol{V} = - \nabla P + \boldsymbol{J} \times \boldsymbol{B}, \qquad (2)$$

where $\boldsymbol{V} = \boldsymbol{V}_e + d_i \boldsymbol{J}$ is the ion velocity and $\boldsymbol{J} = \nabla \times \boldsymbol{B}$. Also, $\nabla \cdot \boldsymbol{B} = 0$ and $\nabla \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t$.

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Our system of equations is closed by a simple energy equation:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{V}\cdot\boldsymbol{\nabla}\right)P = -\Gamma\,\boldsymbol{\nabla}\cdot\boldsymbol{V}P, \tag{3}$$

where $\Gamma = 5/3$ is the ratio of specific heats.

B. 2D equations

Suppose that $\partial/\partial z \equiv 0$. Without loss of generality we can write

$$B = \boldsymbol{\nabla}\psi\times\hat{\boldsymbol{z}} + B_z\hat{\boldsymbol{z}}, \tag{4}$$

and $E_z = -\partial\psi/\partial t$. The z component of Eq. (1) yields

$$\epsilon d_i\left(\frac{\partial}{\partial t} + V_e\cdot\boldsymbol{\nabla}\right)V_{ez} = \left(\frac{\partial}{\partial t} + \boldsymbol{V}_e\cdot\boldsymbol{\nabla}\right)\psi. \tag{5}$$

The z component of the curl of Eq. (1) reduces to

$$\epsilon d_i\left(\frac{\partial}{\partial t} + \boldsymbol{V}_e\cdot\boldsymbol{\nabla} + \boldsymbol{\nabla}\cdot\boldsymbol{V}_e\right)\omega_{ez}$$

$$= \left(\frac{\partial}{\partial t} + \boldsymbol{V}_e\cdot\boldsymbol{\nabla}\right)B_z - [V_{ez},\psi] + \boldsymbol{\nabla}\cdot\boldsymbol{V}_e B_z, \tag{6}$$

where

$$\boldsymbol{\omega}_e = \boldsymbol{\nabla}\times\boldsymbol{V}_e \tag{7}$$

and

$$[A,B] \equiv \boldsymbol{\nabla}A\times\boldsymbol{\nabla}B\cdot\hat{\boldsymbol{z}}. \tag{8}$$

To lowest order in ϵ, Eqs. (5) and (6) can be rearranged to give

$$\left(\frac{\partial}{\partial t} + \boldsymbol{V}\cdot\boldsymbol{\nabla}\right)\psi_e = d_i[\psi,B_z], \tag{9}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{V}\cdot\boldsymbol{\nabla}\right)B_{ez} = [V_z,\psi] + d_i[\boldsymbol{\nabla}^2\psi,\psi] - \boldsymbol{\nabla}\cdot\boldsymbol{V}B_{ez}, \tag{10}$$

respectively. Here,

$$\psi_e = \psi - d_e^2\boldsymbol{\nabla}^2\psi, \tag{11}$$

$$B_{ez} = B_z - d_e^2\boldsymbol{\nabla}^2 B_z, \tag{12}$$

where $d_e = \sqrt{\epsilon}d_i$ is the collisionless electron skin depth.

Equation (2) can be written as

$$\left(\frac{\partial}{\partial t} + \boldsymbol{V}\cdot\boldsymbol{\nabla}\right)\boldsymbol{V}_\perp = -\boldsymbol{\nabla}P - B_z\boldsymbol{\nabla}B_z - \boldsymbol{\nabla}^2\psi\boldsymbol{\nabla}\psi, \tag{13}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{V}\cdot\boldsymbol{\nabla}\right)V_z = [B_z,\psi], \tag{14}$$

where $\boldsymbol{V}_\perp = (V_x, V_y, 0)$.

C. Reduction scheme

Let

$$P = P^{(0)} + B^{(0)}p_1 + p_2, \tag{15}$$

$$B_z = B^{(0)} + b_z, \tag{16}$$

where $P^{(0)}$ and $B^{(0)}$ are uniform constants, p_1, p_2, b_z, ψ, $\boldsymbol{V}$, $\boldsymbol{\nabla}$, and $\partial/\partial t$ are all $O(1)$, and

$$P^{(0)} \gg B^{(0)} \gg 1. \tag{17}$$

The purpose of the above ordering is to make the CA wave propagate much faster than any other wave in the system. When this occurs, the CA wave effectively decouples from these other waves, and can therefore be eliminated from our system of equations.

We expect the rapidly propagating CA wave to maintain approximate force balance within the plasma at all time, i.e.,

$$\nabla P + B_z\nabla B_z + \nabla^2\psi\nabla\psi \simeq 0. \tag{18}$$

Hence, it follows from Eq. (17) that

$$p_1 \simeq -b_z. \tag{19}$$

We also expect the CA wave to make the plasma flow almost incompressible, so that

$$\boldsymbol{V} \simeq \boldsymbol{\nabla}\phi\times\hat{\boldsymbol{z}}. \tag{20}$$

The curl of Eq. (13) yields

$$\frac{\partial U}{\partial t} = [\phi,U] + [\boldsymbol{\nabla}^2\psi,\psi], \tag{21}$$

where $U = \nabla^2\psi$.

Now, to lowest order, Eqs. (3) and (10) become

$$\frac{\partial b_{ez}}{\partial t} = [\phi,b_{ez}] + [V_z,\psi] + d_i[\nabla^2\psi,\psi] - \boldsymbol{\nabla}\cdot\boldsymbol{V}B^{(0)}, \tag{22}$$

$$\frac{\partial p_1}{\partial t} = [\phi,p_1] - \frac{\Gamma P^{(0)}}{B^{(0)}}\nabla\cdot\boldsymbol{V}, \tag{23}$$

where $b_{ez} = b_z - d_e^2\nabla^2 b_z$. We can eliminate $\boldsymbol{\nabla}\cdot\boldsymbol{V}$ from the above two equations, making use of Eq. (19), to give

$$\frac{\partial Z_e}{\partial t} = [\phi,Z_e] + c_\beta[V_z,\psi] + d_\beta[\nabla^2\psi,\psi], \tag{24}$$

where $Z = b_z/c_\beta$, $Z_e = Z - c_\beta^2 d_e^2\nabla^2 Z$, $c_\beta = \sqrt{\beta/(1+\beta)}$, $d_\beta = c_\beta d_i$, and $\beta = \Gamma P^{(0)}/[B^{(0)}]^2$.

D. Reduced equations

Our set of reduced collisionless equations takes the form

$$\frac{\partial\psi_e}{\partial t} = [\phi,\psi_e] + d_\beta[\psi,Z], \tag{25}$$

$$\frac{\partial Z_e}{\partial t} = [\phi,Z_e] + c_\beta[V_z,\psi] + d_\beta[\nabla^2\psi,\psi], \tag{26}$$

$$\frac{\partial U}{\partial t} = [\phi,U] + [\nabla^2\psi,\psi], \tag{27}$$

$$\frac{\partial V_z}{\partial t} = [\phi,V_z] + c_\beta[Z,\psi], \tag{28}$$

where

$$\nabla^2\phi = U, \tag{29}$$

$$\psi_e = \psi - d_e^2\nabla^2\psi, \tag{30}$$

$$Z_e = Z - c_\beta^2 d_e^2\nabla^2 Z, \tag{31}$$

and

$$d_\beta = c_\beta d_i, \tag{32}$$

$$c_\beta = \sqrt{\frac{\beta}{1+\beta}}, \tag{33}$$

$$\beta = \frac{\Gamma P^{(0)}}{[B^{(0)}]^2}. \tag{34}$$

Note that the parameter β measures the strength of the guide field. The large guide-field limit corresponds to $\beta \ll 1$, whereas $\beta \gg 1$ corresponds to the small guide-field limit. Note also that d_i only occurs in our reduced equations in the combination $d_\beta = [\beta/(1+\beta)]^{1/2} d_i$. In the small guide-field limit, $d_\beta \simeq d_i$. However, in the large guide-field limit $d_\beta \simeq \sqrt{\beta} d_i \equiv \rho_s$.

Equations (25)–(28) are generalized versions of the reduced equations presented in Ref. 11 which take electron inertia into account. The linearized form of these equations was first derived by Mirnov, Hegna, and Prager.[12]

E. Zero guide-field limit

In the zero guide-field limit $B^{(0)} \to 0$, which corresponds to $\beta \to \infty$, we adopt the modified ordering

$$P = P^{(0)} + p_1, \tag{35}$$

$$B_z = b_z, \tag{36}$$

where $P^{(0)} \gg 1$ is a uniform constant, and p_1, b_z, ψ, $\boldsymbol{V}$, $\boldsymbol{\nabla}$, and $\partial/\partial t$ are all $O(1)$. As before, the purpose of this ordering scheme is to make the compressional Alfvén wave propagate much faster than any other wave in the system.

From Eq. (3), the above ordering scheme implies that $\boldsymbol{\nabla}\cdot\boldsymbol{V} = 0$. In this limit, the remaining two-fluid equations readily yield Eqs. (25)–(28), with $c_\beta = 1$ and $d_\beta = d_i$.

We conclude that Eqs. (25)–(28) hold in the zero, small, and large guide-field limits, as long as the compressional Alfvén wave propagates more rapidly than any other wave in the system.

III. LARGE Δ' GROWTH RATE

A. Introduction

Let us calculate the linear growth rate of a reconnecting instability from Eqs. (25)–(28) in the limit in which the tearing stability index[13] Δ' is very large.

Let

$$\psi(x,y,t) = -\tfrac{1}{2}x^2 + \tilde{\psi}(x) e^{i(ky+\gamma t)}, \tag{37}$$

$$Z(x,y,t) = \tilde{Z}(x) e^{i(ky+\gamma t)}, \tag{38}$$

$$\phi(x,y,t) = \tilde{\phi}(x) e^{i(ky+\gamma t)}, \tag{39}$$

$$V_z(x,y,t) = \tilde{V}_z(x) e^{i(ky+\gamma t)}, \tag{40}$$

where "$\sim$" denotes a perturbed quantity.

Throughout most of the plasma, the perturbation is governed by the equations of ideal MHD, which yield $(d^2/dx^2 - k^2)\tilde{\psi} = 0$. Ideal MHD breaks down in a thin layer [of thickness $O(d_\beta)$ or $O(d_e)$] centered on $x = 0$.[13] Both d_e and d_β are assumed to be much less than unity. The layer solution must be asymptotically matched to the ideal-MHD solution via the tearing stability index, $\Delta' = [d \ln \tilde{\psi}/dx]_{0-}^{0+}$.

For $x \ll 1$, Eqs. (25)–(28) yield the following linearized layer equations:

$$g\left(1 - d_e^2\frac{d^2}{dx^2}\right)\tilde{\psi} = \mathrm{i}x\tilde{\phi} - \mathrm{i}d_\beta x\tilde{Z}, \tag{41}$$

$$g\left(1 - c_\beta^2 d_e^2\frac{d^2}{dx^2} + \frac{c_\beta^2}{g^2}x^2\right)\tilde{Z} - \mathrm{i}d_\beta x\frac{d^2\tilde{\psi}}{dx^2}, \tag{42}$$

$$g\frac{d^2\tilde{\phi}}{dx^2} = \mathrm{i}x\frac{d^2\tilde{\psi}}{dx^2}, \tag{43}$$

where $g = \gamma/k$.

Let

$$\bar{\phi}(p) = \int_{-\infty}^{+\infty} \tilde{\phi}(x) e^{-\mathrm{i}px}dx, \tag{44}$$

etc. The above Fourier transform should be understood in the generalized sense, since $\phi(x)$ is not square-integrable. The Fourier transformed layer equations take the form

$$g(1 + d_e^2p^2)\bar{\psi} = -\frac{d(\bar{\phi} - d_\beta\bar{Z})}{dp}, \tag{45}$$

$$g\left(1 + c_\beta^2 d_e^2 p^2 - \frac{c_\beta^2}{g^2}\frac{d^2}{dp^2}\right)\bar{Z} = d_\beta\frac{d(p^2\bar{\psi})}{dp}, \tag{46}$$

$$gp^2\bar{\phi} = -\frac{d(p^2\bar{\psi})}{dp}, \tag{47}$$

respectively.

B. Inner layer

In the limit $p \gg c_\beta/g,\ d_\beta^{-1}$, Eqs. (45)–(47) reduce to

$$\frac{d}{dr}\left(\frac{r^2}{1+r^2}\frac{dW}{dr}\right) - Q^2(1 + c_\beta^2 r^2)W = 0, \tag{48}$$

where $r = d_e p$, $W = p^2\bar{\phi}/(1 + c_\beta^2 d_e^2 p^2)$, and $Q = g/d_\beta$.

In the limit $r \to 0$, the asymptotic solution of the above equation takes the form

$$W = W_0[r^{-1-\nu} + \alpha r^\nu + O(r^{1-\nu})], \tag{49}$$

where $\nu = (1/2)(\sqrt{1+4Q^2} - 1)$ and α is determined by the condition that the solution to Eq. (48) be well behaved as $r \to \infty$.

Suppose that $Q \ll 1$, which implies that $0 < \nu \ll 1$. For $r \lesssim 1$, Eq. (48) yields

$$\frac{d}{dr}\left(\frac{r^2}{1+r^2}\frac{dW}{dr}\right) \simeq 0. \tag{50}$$

Integrating, and comparing with Eq. (49) (recalling that $\nu \to 0$), we obtain

$$W = W_0[(r^{-1} - r) + \alpha + O(r^2)] \tag{51}$$

for $r \lesssim 1$.

For $r \gg 1$, Eq. (48) reduces to

$$\frac{d^2W}{dr^2} - Q^2(1 + c_\beta r^2)W \simeq 0, \tag{52}$$

which is a parabolic cylinder equation.[14] The solution which is well behaved as $r \to \infty$ is

$$W \propto U(Q/2c_\beta, \sqrt{2Qc_\beta}r), \tag{53}$$

where $U(a,x)$ is a standard parabolic cylinder function.[14] For $1 \ll r \ll (Qc_\beta)^{-1/2}$, we obtain[14]

$$W = W_1\left[\frac{\Gamma(1/4 + Q/4c_\beta)}{2\sqrt{Qc_\beta}\Gamma(3/4 + Q/4c_\beta)} - r + O(r^2)\right], \tag{54}$$

where $\Gamma(x)$ is standard Gamma function.[14] Finally, comparing with Eq. (51), we find

$$\alpha = Q^{-1}G(Q/c_\beta), \tag{55}$$

where

$$G(x) = \frac{\sqrt{x}}{2}\frac{\Gamma(1/4 + x/4)}{\Gamma(3/4 + x/4)}. \tag{56}$$

Note that $G(x) \to 1$ as $x \to \infty$.

C. Outer layer

Suppose that $x \gg d_e$. In this limit, the real-space layer equations (41)–(43) reduce to

$$\frac{d}{dx}\left[x^2\frac{d}{dx}\left(\frac{g^2 d_\beta^2}{g^2 + c_\beta^2 x^2}\frac{d^2\tilde{\phi}}{dx^2}\right) - (g^2 + x^2)\frac{d\tilde{\phi}}{dx}\right] = 0. \tag{57}$$

Now, we expect the solution to the above equation to match to the ideal-MHD solution at (relatively) large x. Thus,

$$\tilde{\phi} \to \tilde{\phi}_0\left[1 + \frac{2}{\Delta'}\frac{1}{x} + O\left(\frac{1}{x^2}\right)\right] \tag{58}$$

at large x, where Δ' is the standard tearing stability index.[13] Let $Y = g(d\tilde{\phi}/dx)/\tilde{\phi}_0$. It follows that

$$Y \simeq -\frac{2g}{\Delta'}\frac{1}{x^2} \tag{59}$$

at large x. Thus, integrating Eq. (57), we obtain

$$x^2\frac{d}{dx}\left(\frac{g^2 d_\beta^2}{g^2 + c_\beta^2 x^2}\frac{dY}{dx}\right) - (g^2 + x^2)Y = \frac{2g}{\Delta'}. \tag{60}$$

In the large Δ' limit (i.e., $1/\Delta' \to 0$), the above equation reduces to

$$s^2\frac{d}{ds}\left(\frac{1}{1 + c_\beta^2 s^2}\frac{dY}{ds}\right) - Q^2(1 + s^2)Y = 0, \tag{61}$$

where $s = x/g$. Let $V = (1 + c_\beta^2 s^2)^{-1}dY/ds$. The above equation can be transformed into

$$\frac{d}{ds}\left(\frac{s^2}{1+s^2}\frac{dV}{ds}\right) - Q^2(1 + c_\beta^2 s^2)V = 0. \tag{62}$$

Note that this outer layer equation has exactly the same form as the inner layer equation (48). The small s asymptotic solution of Eq. (62) is

$$V = V_0[s^{-1-\nu} + \alpha_1 s^\nu + O(s^{1-\nu})], \tag{63}$$

where α_1 is determined by the condition that V does not blow up as $s \to \infty$. However, since Eq. (62) has exactly the same form as Eq. (48), we immediately deduce that

$$\alpha_1 = \alpha = Q^{-1}G(Q/c_\beta). \tag{64}$$

Fourier transforming Eq. (63), we obtain the asymptotic solution

$$W = W_2\left[\alpha\Gamma(1+\nu)\cos(\nu\pi/2)r^{-1-\nu} - \Gamma(-\nu)\sin(\nu\pi/2) \times\left(\frac{g}{d_e}\right)^{1+2\nu} r^\nu + O(r^{1-\nu})\right] \tag{65}$$

in Fourier space. Comparison with Eq. (49) yields

$$\alpha^2 = \frac{\pi}{2}\frac{g}{d_e} \tag{66}$$

in the limit $\nu \to 0$. It follows from Eq. (55) that

$$g^3 = \frac{2}{\pi}d_e d_\beta^2 G^2(g/c_\beta d_\beta). \tag{67}$$

Our derivation of this expression is valid provided

$$\frac{d_e}{d_\beta} \ll Q \ll 1 \tag{68}$$

and $d_\beta \gg d_e$. These constraints all reduce to the single constraint $\beta \gg m_e/m_i$.

D. Discussion

We have derived the linear dispersion relation (67) from out reduced equations in the large Δ' limit. This dispersion relation yields[15]

$$g = \left(\frac{2}{\pi}\right)^{1/3} d_e^{1/3} d_\beta^{2/3} \tag{69}$$

for $m_e/m_i \ll \beta \ll (m_e/m_i)^{1/4}$, and

$$g = \frac{1}{\sqrt{2\pi}}\frac{\Gamma(1/4)}{\Gamma(3/4)}\sqrt{d_e d_i} \tag{70}$$

for $(m_e/m_i)^{1/4} \ll \beta$. Note that our derivation of Eq. (67) is valid for all β values significantly above m_e/m_i.

The dispersion relation (67) and the growth rate (70) were first obtained by Mirnov, Hegna, and Prager.[12] However, the derivation presented in Ref. 12 is only valid for β in the range $(m_e/m_i) \ll \beta \ll 1$. The improved derivation presented in this paper removes the upper constraint on β, and, thereby, confirms that Eq. (67) is valid in both the large and small guide-field limits.

IV. SMALL Δ' GROWTH RATE

Let us now calculate the linear growth rate of a reconnecting instability from Eqs. (25)–(28) in the limit in which the tearing stability index Δ' is relatively small.

In the small Δ' limit, we expect the tearing layer to be localized on the d_e length scale and to be governed solely by electron physics. Taking Eqs. (45) and (46) and neglecting the ion terms (i.e., the terms involving $\bar{\phi}$ and $d^2\bar{Z}/dp^2$), we obtain

$$\frac{d}{dr}\left(\frac{r^2}{1+r^2}\frac{d\bar{Z}}{dr}\right) - Q^2(1+c_\beta^2 r^2)\bar{Z} = 0 \tag{71}$$

and

$$\bar{\psi} = \frac{d_\beta}{1+d_e^2 p^2}\frac{d\bar{Z}}{dp}, \tag{72}$$

where $r=d_e p$ and $Q=g/d_\beta$. The solution to Eq. (71) which is well behaved as $r\to\infty$ has the small r expansion,

$$\bar{Z} = \bar{Z}_0[r^{-1} + \alpha + O(r)], \tag{73}$$

assuming that $Q \ll 1$. Here, α is given by Eq. (55) since Eq. (71) has exactly the same form as Eq. (48). Inverse Fourier transforming, we obtain

$$\tilde{Z} = \tilde{Z}_0\left[\frac{\pi}{2d_e\alpha} + \frac{1}{x} + O\left(\frac{1}{x^2}\right)\right] \tag{74}$$

and

$$\tilde{\psi} \sim x\tilde{Z} = \tilde{Z}_0\left[\frac{\pi}{d_e\alpha}\frac{x}{2} + 1 + O\left(\frac{1}{x}\right)\right]. \tag{75}$$

It follows by matching to the standard ideal-MHD solution,

$$\tilde{\psi} = \tilde{\psi}_0\left[\frac{\Delta'}{2}x + 1 + O\left(\frac{1}{x}\right)\right], \tag{76}$$

that $\Delta' = \pi/d_e\alpha$, or

$$\Delta' = \frac{\pi}{d_e d_\beta}\frac{g}{G(g/c_\beta d_\beta)}. \tag{77}$$

Hence, we conclude that[15]

$$g = \frac{d_\beta}{\pi}\Delta' d_e \tag{78}$$

for $\beta \ll (\Delta' d_e)^2$, and

$$g = \left(\frac{\Gamma(1/4)}{2\pi\Gamma(3/4)}\right)^2 d_i(\Delta' d_e)^2 \tag{79}$$

for $\beta \gg (\Delta' d_e)^2$.

The dispersion relation (77) was first obtained by Mirnov, Hegna, and Prager.[12]

V. GENERAL DISPERSION RELATION

Let us now derive a general dispersion relation from Eqs. (25)–(28) which holds for large, small, and intermediate values of Δ'.

For finite Δ', Eqs. (60) and (62) yield

$$s^2\frac{d}{ds}\left(\frac{1}{1+c_\beta^2 s^2}\frac{dY}{ds}\right) - Q^2(1+s^2)Y = -\hat{\lambda}_H Q, \tag{80}$$

$$\frac{d}{ds}\left(\frac{s^2}{1+s^2}\frac{dV}{ds}\right) - Q^2(1+c_\beta^2 s^2)V = -\hat{\lambda}_H Q\frac{d}{ds}\left(\frac{1}{1+s^2}\right), \tag{81}$$

where $s=x/g$, $\lambda_H=-\pi/\Delta'$, and $\hat{\lambda}_H=(\lambda_H/d_\beta)(2/\pi)$. Note, from Eq. (58), that $Y(s)$ is subject to the constraint

$$\int_0^\infty Y(s)ds = 1. \tag{82}$$

Our basic expansion parameter is $H=G(Q/c_\beta)/Q$ [see Eq. (56)]. Note that $H \gg 1$ provided $Q \ll 1$. It can easily be verified from the following analysis that $V \sim O(Q^2)$ and $\hat{\lambda}_H \lesssim O(Q/H)$. It follows that in Eqs. (80) and (81) the inhomogenous term on the right-hand side can be neglected for $s \gg H^{-1}$, whereas the second term on the left-hand side can be neglected for $s \ll H$.

In the limit $s \ll H$, Eq. (81) reduces to

$$\frac{d}{ds}\left(\frac{s^2}{1+s^2}\frac{dV}{ds}\right) \simeq -\hat{\lambda}_H Q\frac{d}{ds}\left(\frac{1}{1+s^2}\right), \tag{83}$$

which can be solved to give

$$V \simeq -\frac{a_0 - \hat{\lambda}_H Q}{s} + a_1 + a_0 s, \tag{84}$$

where $a_0(Q,c_\beta)$ and $a_1(Q,c_\beta)$ are constants of integration. Note that the first two terms on the right-hand side of the above expression are negligible for $s \gg H^{-1}$, and *vice versa*.

In the limit $s \ll H^{-1}$, the above expression can be asymptotically matched to the inner layer solution described in Sec. III B. In real space, this solution yields (for $Q \ll 1$)

$$V \simeq V_0\left[\frac{2}{\pi}\frac{d_e}{d_\beta}\frac{G(Q/c_\beta)}{Q^2}\frac{1}{s} + 1 + O(s)\right]. \tag{85}$$

It therefore follows that

$$\frac{a_0 - \hat{\lambda}_H Q}{a_1} = -\frac{2}{\pi}\frac{d_e}{d_\beta}\frac{G(Q/c_\beta)}{Q^2}. \tag{86}$$

In the limit $s \gg H^{-1}$, Eq. (81) reduces to

$$\frac{d^2V}{ds^2} - Q^2(1+c_\beta^2 s^2)V \simeq 0. \tag{87}$$

This is a parabolic cylinder equation.[14] The solution which is well behaved as $s\to\infty$ has the expansion

$$V \sim V_0\left[1 - \frac{Q}{G(Q/c_\beta)}s + O(s^2)\right] \tag{88}$$

for $s \ll H$. Asymptotic matching to Eq. (84) yields

$$\frac{a_0}{a_1} = -\frac{Q}{G(Q/c_\beta)}. \tag{89}$$

Note that for $s \gg H^{-1}$ we can write

$$V \simeq a_1 \frac{U(q/2c_\beta, \sqrt{2Qc_\beta}s)}{U(Q/2c_\beta, 0)}, \tag{90}$$

where $U(a,x)$ is a parabolic cylinder function.[14]

Dividing Eq. (80) by $1+s^2$ and then integrating from $s=0$ to ∞, we obtain

$$\int_0^\infty \frac{s^2}{1+s^2}\frac{dV}{ds}ds = Q^2 - \hat{\lambda}_H Q \frac{\pi}{2}, \tag{91}$$

where use has been made of Eq. (82). The dominant contribution to the integral in the above expression comes from the region $s \gg 1$. Furthermore, $\hat{\lambda}_H \lesssim O(Q/H)$, so the second term on the right-hand side is negligible. Hence, it follows from (90) that

$$a_1 \left[\frac{U(Q/2c_\beta, \sqrt{2Qc_\beta}s)}{U(Q/2c_\beta, 0)} \right]_{s=0}^{s=\infty} \simeq Q^2 \tag{92}$$

or

$$a_1 \simeq -Q^2. \tag{93}$$

Equations (86), (89), and (93) can be combined to give

$$\frac{Q}{G(Q/c_\beta)} - \frac{\hat{\lambda}_H}{Q} = \frac{2}{\pi}\frac{d_e}{d_\beta}\frac{G(Q/c_\beta)}{Q^2}. \tag{94}$$

Hence, our general dispersion relation takes the form

$$\lambda_H = \frac{\pi}{2}\frac{g^2}{d_\beta G(g/c_\beta d_\beta)} - \frac{d_e d_\beta G(g/c_\beta d_\beta)}{g}, \tag{95}$$

where $\lambda_H = -\pi\,\Delta'$. This result is valid provided $g \ll d_\beta$, which implies that $\lambda_H \ll d_\beta$, and $\beta \gg m_e/m_i$. The above dispersion relation is consistent with the dispersion relations (67) and (77) which we previously derived in the large and small Δ' limits, respectively. Equation (95) was first obtained by Mirnov, Hegna, and Prager[12] using a heuristic argument. We, on the other hand, have derived this result in a rigorous fashion. In the limit $\beta \ll 1$, the above expression reduces to a dispersion relation previously obtained by Porcelli.[15]

VI. SUMMARY

We have derived a set of reduced equations (25)–(28), governing nonlinear, 2D, two-fluid, collisionless magnetic reconnection with (essentially) arbitrary guide field. These equations represent an improvement on the nonlinear reduced equations previously used to investigate collisionless reconnection,[8,9] which are only valid in the large guide-field limit. The linearized version of our set of equations was first derived by Mirnov, Hegna, and Prager.[12]

Using our equations, we have calculated the linear growth rate of a highly unstable, spontaneously reconnecting, plasma instability. Our expression for the growth rate, Eq. (67), is valid in both the large and small guide-field limits. This expression was previously obtained by Mirnov, Hegna, and Prager.[12] However, the derivation given in Ref. 12 is only valid in the large guide-field limit. Interestingly enough (as also remarked by Mirnov, Hegna, and Prager), expression (67) indicates that the guide field ceases to have any effect on the linear growth rate once the plasma β calculated with the guide field significantly exceeds the rather small critical value $(m_e/m_i)^{1/4} \simeq 0.16$ (for hydrogen).

We have also derived the general linear dispersion relation, (95) of a spontaneously reconnecting plasma instability. This result was previously obtained by Mirnov, Hegna, and Prager[12] using a heuristic argument.

In future work, we intend to investigate the nonlinear properties of our equations.

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[1] F. L. Waelbroeck, Phys. Fluids B **1**, 2372 (1989).
[2] K. Shibata, Adv. Space Res. **17**, 9 (1996).
[3] V. M. Vasyliunas, Rev. Geophys. Space Phys. **13**, 303 (1975).
[4] D. Biskamp, Phys. Fluids **29**, 1520 (1986).
[5] B. Coppi, Phys. Rev. Lett. **11**, 226 (1964).
[6] M. Ottaviani and F. Porcelli, Phys. Rev. Lett. **71**, 3802 (1993).
[7] H. R. Strauss, Phys. Fluids **20**, 1357 (1977).
[8] R. D. Hazeltine, M. Kotschenreuther, and P. G. Morrison, Phys. Fluids **28**, 2466 (1985).
[9] D. Grasso, F. Pegoraro, F. Porcelli, and F. Califano, Plasma Phys. Controlled Fusion **41**, 1497 (1999).
[10] S. I. Braginskii, in *Reviews of Plasma Physics* (Consultants Bureau, New York, 1965), Vol. 1, p. 205.
[11] R. Fitzpatrick, Phys. Plasmas (submitted).
[12] V. V. Mirnov, C. C. Hegna, and S. C. Prager, Bull. Am. Phys. Soc. **47**, 87 (2001); Plasma Phys. Rep. **29**, 566 (2003); Phys. Plasmas (to be published).
[13] H. P. Furth, J. Killeen, and M. N. Rosenbluth, Phys. Fluids **6**, 459 (1963).
[14] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York 1965).
[15] F. Procelli, Phys. Rev. Lett. **66**, 425 (1991).