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ON THE "$1\frac{1}{2}$-D" EVOLUTION OF TOKAMAK PLASMAS IN THE CASE OF LARGE ASPECT RATIO

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Abstract—The theory of the resistive evolution of an axisymmetric toroidal plasma hinges on calculating self-consistent conditions on the plasma velocity. These conditions ensure that the plasma proceeds along a sequence of equilibrium states, and at finite aspect ratio involve computing the averages of certain equilibrium quantities around flux surfaces at every time step. In this paper we utilize the established representation of a large aspect ratio Tokamak equilibrium to show that for this case the flux-surface averages can be evaluated analytically. The resulting transport equations, which can be developed to high order in inverse aspect ratio, are explicitly 1-D while containing effects of both toroidicity and shaping.

1. INTRODUCTION

THE EVOLUTION of plasma on diffusive time scales can be modelled, in principle, by the simple stratagem of neglecting the inertial term in the equation of motion. This has the merit of "filtering out" Alfvénic phenomena but the disadvantage is that there remains no explicit way of advancing the plasma velocity $\mathbf{v}$ in time. However, as shown by GRAD and HOGAN (1970) and developed by TAYLOR (private communication, 1975), there remain implicit constraints on $\mathbf{v}$ that are imposed by the requirement that any evolution of pressure or magnetic induction implies a $\mathbf{v}$ such that force balance or equilibrium is maintained at all times. In fact, in axisymmetric geometry these constraints can be written down and used to formulate a set of self-consistent (albeit non-standard) evolution equations (TAYLOR, private communication, 1975). The constraints take the form of relationships between flux-surface averages of equilibrium quantities, and the system has become known as the "$1\frac{1}{2}$-D" model. However, as this system involves the solution of a Grad–Shafranov equilibrium problem at each time step, followed by the evaluation of the flux-surface averages, the computation is complex and demanding in computer time.

In this paper we formulate the $1\frac{1}{2}$-D problem using the large aspect ratio Tokamak representation of GREENE *et al.* (1971). An aspect ratio ($\varepsilon = a/R \ll 1$) expansion is used, and we find that the flux-surface averaging order by order in ε can be performed analytically. The resulting evolution equations for the pressure, toroidal and poloidal magnetic fields etc., are then strictly 1-D and lead to a very efficient computational simulation. The effects of toroidicity and shaping (ellipticity and triangularity) are all included to the relevant order in ε.

Section 2 of this paper reviews the equilibrium formulation that underlies the method, while Section 3 describes the diffusion model and derives the 1-D evolution equations that govern the system. A discussion of the boundary conditions in Section 4 is followed by a summary and conclusions in Section 5.

2. EQUILIBRIUM FORMULATION

We base our equilibrium formulation on the description given by GREENE *et al.* (1971) and CONNOR and HASTIE (1985). Thus the equilibrium magnetic field is taken to be of the form

$$\mathbf{B} = R_0 B_0 [f(r)\nabla\phi \wedge \nabla r + g(r)\nabla\phi], \tag{2.1}$$

where R_0 is the distance from the symmetry axis to the geometric centre of the (nearly circular) plasma boundary, and B_0 is the toroidal magnetic field strength at this point in the absence of plasma. The r, θ, ϕ coordinates are a non-orthogonal right-handed set (CONNOR and HASTIE, 1985), with ϕ the angle around the symmetry axis, r constant on a magnetic surface, and θ a poloidal angle-like variable chosen so that the field lines are "straight". In these coordinates, the volume element is

$$\mathrm{d}\tau = r\frac{R^2}{R_0}\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi. \tag{2.2}$$

The operator $\mathbf{B}\cdot\nabla$ is given by

$$\mathbf{B}\cdot\nabla = \frac{R_0^2}{R^2}\frac{B_0 f}{r}\left[\frac{\partial}{\partial\theta} + q\frac{\partial}{\partial\phi}\right], \tag{2.3}$$

where $R = |\nabla\phi|^{-1}$ is the distance from the symmetry axis, and q is the safety factor defined below. In (r, θ, ϕ) coordinates the equilibrium Grad–Shafranov equation takes the form (CONNOR *et al.*, 1988)

$$\frac{1}{r}\frac{\partial}{\partial r}(rf|\nabla r|^2) + \frac{1}{r}\frac{\partial}{\partial\theta}(rf\nabla r\cdot\nabla\theta) + \frac{1}{f}\left(gg' + \frac{p'}{B_0^2}\frac{R^2}{R_0^2}\right) = 0, \tag{2.4}$$

where $'$ denotes $\mathrm{d}/\mathrm{d}r$ and the various magnetic and current fluxes are given by:

(i) toroidal flux $\chi(r)$

$$\chi(r) = 2\pi B_0 \int_0^r rg(r)\,\mathrm{d}r, \tag{2.5}$$

(ii) poloidal flux $\Psi(r)$

$$\Psi(r) = 2\pi R_0 B_0 \int_0^r f(r)\,\mathrm{d}r, \tag{2.6}$$

(iii) toroidal current flux $I(r)$

$$I(r) = 2\pi r B_0 f\langle|\nabla r|^2\rangle, \tag{2.7}$$

where $\langle X \rangle \equiv \frac{1}{2\pi}\oint X\,\mathrm{d}\theta$, and

(iv) safety factor $q(r)$

$$q(r) = \frac{\mathrm{d}\chi}{\mathrm{d}\Psi} = \frac{rg}{R_0 f}. \tag{2.8}$$

Introducing the large aspect ratio, and weak shaping, expansions we write

$$g = 1+g_2+g_4+\cdots,\quad f=f_1+f_3+f_4+\cdots,\quad p=p_2+p_4+\cdots, \tag{2.9}$$

where the subscripts indicate the order of magnitude in ε of each term. In CONNOR and HASTIE (1985) and CONNOR (1988), the metric elements have been calculated to $O(\varepsilon)$ with the retention of the ε^2 contribution to $\langle|\nabla r|^2\rangle$, with the results

$$|\nabla r|^2 = 1-2\Delta'\cos\theta+2\sum_n S_n'\cos n\theta+\frac{1}{2}\Delta'^2+\frac{3}{4}\frac{r^2}{R_0^2}+\frac{\Delta}{R_0}+\frac{1}{2}\sum_n[(S_n')^2+(n^2-1)S_n^2/r^2]+\cdots, \tag{2.10}$$

$$\nabla r\cdot\nabla\theta = \frac{1}{r}\left\{\left(r\Delta''+\Delta'+\frac{r}{R}\right)\sin\theta-\frac{1}{n}\left(rS_n''+S_n'+(n^2-1)\frac{S_n}{r}\right)\sin n\theta+\cdots\right\}, \tag{2.11}$$

$$\left(\frac{R}{R_0}\right)^2 = 1-\frac{2r}{R_0}\cos\theta-\frac{2\Delta}{R_0}-\frac{r}{R_0}\Delta'-\frac{1}{2}\frac{r^2}{R_0^2}+\cdots. \tag{2.12}$$

The value of r at the plasma boundary is related to the radius a of this nearly circular boundary by

$$r_a = a\left[1+\frac{1}{8}\frac{a^2}{R_0^2}-\frac{1}{2}\sum_n(n-1)\frac{S_n^2}{a^2}+\cdots\right]. \tag{2.13}$$

In these equations, the quantity $\Delta = \Delta(r)$ represents the toroidally-induced shift of the equilibrium magnetic surfaces, and the $S_n(r)$ describe the harmonic distortion of the magnetic surfaces (cf. Appendix A). Equations governing these quantities are obtained by introducing the expansions (2.9)–(2.12) into the equilibrium Grad–Shafranov equation (2.4) and decomposing by harmonic in θ. Thus in leading order, one obtains the cylindrical pressure balance equation

$$\frac{p_2'}{B_0^2}+g_2'+\frac{f_1}{r}(rf_1)' = 0. \tag{2.14}$$

At first order in ε, equations are obtained for $\Delta(r)$ (from the $\cos\theta$ harmonic) and for the externally-imposed shaping of the magnetic surfaces:

$$\Delta''+\Delta'\left(\frac{2(rf_1)'}{rf_1}-\frac{1}{r}\right)-\frac{1}{R_0}+\frac{2rp'}{R_0f_1^2B_0^2}=0, \tag{2.15}$$

$$S_n''+S_n'\left(\frac{2(rf_1)'}{rf_1}-\frac{1}{r}\right)-(n^2-1)\frac{S_n}{r^2}=0. \tag{2.16}$$

At second order in ε, the second-order correction to the pressure balance is obtained from the θ-independent terms in (2.4)

$$\begin{aligned}\frac{p_4'}{B_0^2}+g_4'+\frac{1}{r^2}(r^2f_1f_3)' &=\frac{f_1^2}{r}\left\{g_2-\frac{9}{4}\frac{r^2}{R_0^2}-\frac{\Delta}{R_0}-\frac{2r\Delta'}{R_0}+\frac{1}{2}\Delta'^2\right.\\ &\left.+\sum_n\left[\frac{1}{2}(n^2-1)\frac{S_n^2}{r^2}-2(n^2-1)S_n'\frac{S_n}{r}+\frac{1}{2}S_n'^2\right]\right\}\\ &+f_1f_1'\left\{g_2-\frac{3}{4}\frac{r^2}{R_0^2}-\frac{\Delta}{R_0}+\frac{3}{4}\Delta'^2+\sum_n\left[-\frac{1}{2}(n^2-1)\frac{S_n^2}{r^2}+\frac{3}{2}S_n'^2\right]\right\}\\ &+p_2'\left\{g_2+\frac{1}{2}\frac{r^2}{R_0^2}+\frac{2\Delta}{R_0}+\frac{3r\Delta'}{R_0}\right\}\end{aligned} \tag{2.17}$$

while a correction $\Delta_2(r)$ to the Shafranov shift is obtained from the $\cos\theta$ terms. This correction is driven by coupling between adjacent shaping harmonics, and also by coupling of ellipticity to toroidal effects. We find

$$\begin{aligned}\Delta_2''+\Delta_2'\left(\frac{2(rf_1)'}{rf_1}-\frac{1}{r}\right) &=3\Delta'S_2'\left(\frac{(rf_1)'}{rf_1}-\frac{1}{r}\right)-\frac{3\Delta'S_2}{r^2}+\frac{3p_2'r}{R_0B_0^2f_1^2}\left(S_2'+\frac{S_2}{r}\right)\\ &-\frac{S_2'}{R_0}+\frac{S_2S_3}{r^2}\left(\frac{2(rf_1)'}{rf_1}+\frac{10}{r}\right)+\frac{10S_2'S_3}{r^2}+\frac{5S_3'S_2}{r^2}-S_2'S_3'\left(\frac{3(rf_1)'}{rf_1}-\frac{5}{r}\right),\end{aligned} \tag{2.18}$$

where we have neglected S_n for $n>3$ (i.e. we retain only S_2 and S_3, the ellipticity and triangularity, respectively).

The $\cos 2\theta$ and $\cos 3\theta$ harmonics yield next-order contributions to the ellipticity and triangularity, S_{22} and S_{32}:

$$\begin{aligned}S_{22}''+S_{22}'\left(\frac{2(rf_1)'}{rf_1}-\frac{1}{r}\right)-\frac{3S_{22}}{r^2} &=\frac{3}{2}\Delta'^2\frac{(rf_1)'}{rf_1}-\frac{\Delta'}{R_0}-\frac{r}{2R_0^2}+\frac{p_2'r}{R_0B_0^2f_1^2}\left(3\Delta'-\frac{r}{2R_0}-3S_3'-\frac{4S_3}{r}\right)\\ &-\Delta'S_3'\left(\frac{3(rf_1)'}{rf_1}-\frac{4}{r}\right)+\frac{8\Delta'S_3}{r^2}+\frac{S_3'}{R_0},\end{aligned} \tag{2.19}$$

$$S_{32}''+S_{32}'\left(\frac{2(rf_1)'}{rf_1}-\frac{1}{r}\right)-\frac{8S_{32}}{r^2}=-\Delta'S_2'\left(\frac{3(rf_1)'}{rf_1}+\frac{1}{r}\right)+\frac{rp_2'}{R_0B_0^2f_1^2}\left(\frac{S_2}{r}-3S_2'\right)$$
$$+\frac{3\Delta'S_2}{r^2}+\frac{S_2'}{R_0}. \quad (2.20)$$

These higher-order equations (2.18)–(2.20) are needed if we wish to study the effect of shaping on localized (MERCIER, 1960) and resistive-interchange modes (GLASSER *et al.*, 1975). We derive these instability criteria in Appendix B. We note in passing that the derivation of equations (2.18)–(2.20) requires knowledge of the harmonic content of $|\nabla r|^2$, $|\nabla r\cdot\nabla\theta|$ and R^2 to ε^2 order, not displayed in equations (2.10)–(2.12); this information is recorded in Appendix C.

Equations (2.14)–(2.20) provide a set of constraints which a resistively-evolving equilibrium must always satisfy. In the next section we go on to describe the equations that govern this evolution, again expanding in inverse aspect ratio, to derive the $1\frac{1}{2}$-D model in this case.

3. THE EVOLUTION EQUATIONS

The equations governing the resistive evolution of, and transport in, a toroidal equilibrium are the induction equation

$$\frac{\partial \mathbf{B}}{\partial t}=\nabla\wedge(\mathbf{V}\wedge\mathbf{B})-\nabla\wedge[\boldsymbol{\eta}\cdot(\nabla\wedge\mathbf{B})], \quad (3.1)$$

and equations describing heat and particle transport. For simplicity, we have taken a simple Ohm's law with anisotropic resistivity. We also ignore density evolution [assuming $n_e=n_e(r)$ is time independent] and describe energy transport in a single-fluid description with $T=\frac{1}{2}(T_e+T_i)$. Thus,

$$3n_e\left[\frac{\partial T}{\partial t}+\mathbf{V}\cdot\nabla T\right]+2n_eT(\nabla\cdot\mathbf{V})=\mathbf{E}\cdot\mathbf{J}-\nabla\cdot\mathbf{q}, \quad (3.2)$$

where the heat flux q is given by

$$\mathbf{q}=\kappa_\parallel\nabla_\parallel T+\kappa_\perp\nabla_\perp T, \quad \text{with } \kappa=\kappa_e+\kappa_i. \quad (3.3)$$

Now, introducing the representation (2.1) for the magnetic field in (r,θ,ϕ) coordinates into equation (3.1), we then express the velocity as

$$\mathbf{V}=V_1\nabla r+V_2\nabla\phi\wedge\nabla r+V_3\nabla\phi, \quad (3.4)$$

and note that in an axisymmetric system the poloidal component of (3.1) can be integrated once to give

$$\frac{\partial\psi}{\partial t}+V_1f|\nabla r|^2+\eta_\perp\frac{p'}{B^2}f|\nabla r|^2+\frac{\eta_\parallel g}{f}\left(g'+\frac{gp'}{B^2}\right)=C, \quad (3.5)$$

where $\psi = \int_0^r f\,\mathrm{d}r = \Psi(2\pi R_0 B_0)^{-1}$ and $C = \dot{\psi}(r_a)$ is a measure of the applied loop volts. To evaluate the toroidal $(\nabla\phi\cdot)$ component of (3.1) we need to make use of various geometrical relationships such as

$$\nabla r \wedge \nabla\phi = \frac{r}{R_0}[(\nabla r\cdot\nabla\theta)\nabla r - |\nabla r|^2\nabla\theta], \tag{3.6}$$

(note that as we have a non-orthogonal coordinate system $\nabla r\cdot\nabla\theta \neq 0$). We then find

$$\frac{\partial g}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r|\nabla r|^2\alpha) = \frac{\partial}{\partial\theta}\left\{\frac{R_0}{rR^2}(fV_3 - gV_2) - \frac{(\nabla r\cdot\nabla\theta)\alpha}{R^2}\right\}, \tag{3.7}$$

where

$$\alpha = V_1 g - \eta_\parallel\frac{\partial g}{\partial r} + (\eta_\perp - \eta_\parallel)\frac{g}{B^2}\frac{\partial p}{\partial r}. \tag{3.8}$$

The $1\frac{1}{2}$-D scheme is now to regard (3.5) as providing an explicit formula for V_1 which may be inserted into (3.7). The flows V_2 and V_3 can be eliminated from (3.7) by taking a flux-surface average, i.e. by operating with $(1/2\pi)\oint\mathrm{d}\theta$. On performing these operations there is a cancellation of the perpendicular resistivity terms to give

$$\dot{g} + g'\left\langle\frac{\partial r}{\partial t}\right\rangle = \frac{1}{r}\frac{\partial}{\partial r}[\eta_\parallel r g'\langle|\nabla r|^2\rangle] - \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{\eta_\parallel}{f}g\frac{\partial}{\partial r}(r\langle|\nabla r|^2\rangle f)\right]$$
$$- \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{rg}{f}\left(C - \dot{\psi} - f\left\langle\frac{\partial r}{\partial t}\right\rangle\right)\right], \tag{3.9}$$

where $\langle X\rangle \equiv (1/2\pi)\oint X\,\mathrm{d}\theta$, and $\dot{g} = (\partial g/\partial t)_r$ etc.

The energy equation (3.2) is processed in analogous fashion. Again we employ equation (3.5) to eliminate the radial flow and take a poloidal average to annihilate terms involving the poloidal flow. The result is

$$n_e\dot{T}\left\langle\frac{R^2}{R_0^2}\right\rangle + \frac{n_e T'}{f}\left\langle\frac{R^2}{R_0^2}\left[C - \dot{\psi} - f\frac{\partial r}{\partial t} - \eta_\perp\frac{p'f}{B^2}|\nabla r|^2 + \eta_\parallel g\frac{j_\parallel}{B}\right]\right\rangle$$
$$+ \frac{2}{3}n_e T\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{f}\left\langle\frac{R^2}{R_0^2}\left[C - \dot{\psi} - f\frac{\partial r}{\partial t} - \eta_\perp\frac{p'}{B^2}f|\nabla r|^2 + \eta_\parallel g\frac{j_\parallel}{B}\right]\right\rangle\right)$$
$$= \frac{1}{3}\eta_\parallel\left\{\left[\frac{B_0}{r}\frac{\partial}{\partial r}(rf\langle|\nabla r|^2\rangle)\right]^2 + \langle|\nabla r|^2\rangle B_0^2 g'^2 + \frac{p'^2}{B_0^2 f^2}\left[\left\langle\frac{R^4}{R_0^4}\right\rangle - \left\langle\frac{R^2}{R_0^2}\right\rangle^2\right]\right.$$
$$\left. - p'^2\left\langle\frac{R^2}{R_0^2}\frac{|\nabla r|^2}{B^2}\right\rangle\right\} + \frac{1}{3r}\frac{\partial}{\partial r}\left(\left\langle\chi_\perp|\nabla r|^2\frac{R^2}{R_0^2}\right\rangle rnT'\right) + \frac{1}{3}\eta_\perp\left\langle\frac{R^2}{R_0^2}\frac{|\nabla r|^2}{B^2}\right\rangle p'^2. \tag{3.10}$$

Equations (3.9) and (3.10) are exact evolution equations that form the basis of any finite aspect ratio $1\frac{1}{2}$-D simulation. As remarked in the Introduction, a potential difficulty is that the flux-surface averages of (3.9) and (3.10) have to be computed and these themselves involve the changing equilibrium. We now introduce the aspect ratio and shaping expansions into (3.9) and (3.10) and show that the flux-surface averages can be evaluated analytically.

Evaluating $\partial r/\partial t$ directly (cf. Appendix A) yields the result

$$\frac{\partial r}{\partial t} = -\dot{\Delta}\cos\theta + \sum_n \dot{S}_n \cos n\theta + O(\varepsilon^2). \tag{3.11}$$

Thus, in leading order equation (3.9), which at first sight appears to be an evolution equation for g, gives

$$\frac{\partial}{\partial r}\left\{\frac{\eta_\parallel}{f_1}\frac{\partial}{\partial r}(rf_1) + \frac{r}{f_1}(C - \dot{\psi}_1)\right\} = 0, \tag{3.12}$$

which can be rearranged and recognized as a cylindrical diffusion equation for $f_1(r, t)$:

$$\frac{\partial f_1}{\partial t} = \frac{\partial}{\partial r}\left(\frac{\eta_\parallel}{r}\frac{\partial}{\partial r}(rf_1)\right). \tag{3.13}$$

At second order, equation (3.9) can be processed in a similar manner to give the evolution of $f_3(r, t)$;

$$\frac{\partial f_3}{\partial t} = \frac{\partial}{\partial r}\left(\frac{\eta_\parallel}{r}\frac{\partial}{\partial r}(rf_3)\right) + H_3(r, t), \tag{3.14}$$

where

$$H_3(r, t) = \frac{\partial}{\partial r}\left(\frac{\eta_\parallel}{r}\frac{\partial}{\partial r}[rf_1\langle|\nabla r|^2\rangle_2]\right) - \frac{\partial}{\partial r}\left(\eta_\parallel f_1 \frac{\partial g_2}{\partial r}\right) + \frac{\partial}{\partial r}\left[\frac{f_1}{r}\int_0^r \dot{g}_2 r\,\mathrm{d}r\right], \tag{3.15}$$

and $\langle|\nabla r|^2\rangle_2$ represents the $O(\varepsilon^2)$ contribution to $\langle|\nabla r|^2\rangle$, which we calculate from (2.10) giving

$$\langle|\nabla r|^2\rangle_2 = \frac{1}{2}\Delta'^2 + \frac{3}{4}\frac{r^2}{R_0^2} + \frac{\Delta}{R_0} + \frac{1}{2}\left[(S_2')^2 + 3\frac{S_2^2}{r^2} + (S_3')^2 + \frac{8S_3^2}{r^2}\right] \tag{3.16}$$

when only elliptic (S_2) and triangular (S_3) distortions of magnetic surfaces are retained.

In the presence of such shaping terms the next non-trivial order in the expansion of equation (3.9) yields an analogous equation for f_4:

$$\frac{\partial f_4}{\partial t} = \frac{\partial}{\partial r}\left(\frac{\eta_\parallel}{r}\frac{\partial}{\partial r}(rf_4)\right) + H_4(r,t) \tag{3.17}$$

where

$$H_4(r,t) = \frac{\partial}{\partial r}\left(\frac{\eta_\parallel}{r}\frac{\partial}{\partial r}[rf_1\langle|\nabla r|^2\rangle_3]\right). \tag{3.18}$$

For a circular cross-section equilibrium $\langle|\nabla r|^2\rangle_3$, the $O(\varepsilon^3)$ contribution to the metric coefficient $\langle|\nabla r|^2\rangle$ vanishes and $f_4 \equiv 0$. However, in the presence of plasma shaping [with $S_n/r \sim O(\varepsilon)$], f_4 is driven by the coupling of adjacent shaping harmonics $O(\varepsilon S_n S_{n+1})$ and by the coupling of ellipticity to toroidal effects $O(\varepsilon^2 S_2)$. With elliptic and triangular shaping, $\langle|\nabla r|^2\rangle_3$ has been evaluated with the aid of REDUCE (HEARN, 1987), with the result

$$\begin{aligned}\langle|\nabla r|^2\rangle_3 = \Delta'\Delta_2' + \frac{\Delta_2}{R_0} + \left[S_2'S_{22}' + 3\frac{S_2S_{22}}{r^2} + S_3'S_{32}' + 8\frac{S_3S_{32}}{r^2}\right] \\ + \left[-\frac{S_2 r}{R^2} + \frac{3}{4}\Delta'^2\left(\frac{S_2}{r} + S_2'\right) - \frac{1}{2}S_2'\frac{r}{R_0}\Delta'\right] \\ + \left[\frac{S_2S_3}{rR_0} + \frac{1}{2}\frac{S_2'S_3'r}{R_0} - \frac{\Delta'S_2}{r}\left(\frac{S_3}{r} + \frac{1}{2}S_3'\right) - \frac{3}{2}\Delta'S_2'\left(\frac{2S_3}{r} + S_3'\right)\right].\end{aligned} \tag{3.19}$$

Because H_4 contains no dependence on f_3, the equations for f_3 and f_4 may simply be added in practice to obtain a single equation for $\hat{f}_3 = f_3 + f_4$ of the form

$$\frac{\partial \hat{f}_3}{\partial t} = \frac{\partial}{\partial r}\left(\frac{\eta_\parallel}{r}\frac{\partial}{\partial r}(r\hat{f}_3)\right) + \hat{H}_3(r,t) \tag{3.20}$$

where $\hat{H}_3 = H_3 + H_4$.

Returning to equation (3.15), we note that H_3 contains both $g_2(r)$ and its time derivative $\dot{g}_2$. These quantities are obtained from the pressure-balance relation (2.15) and its time derivative

$$\dot{g}_2' + \frac{\dot{p}_2'}{B_0^2} + \frac{\dot{f}_1}{r}(rf_1)' + \frac{f_1}{r}(r\dot{f}_1)' = 0. \tag{3.21}$$

An expression for $\dot{g}_2(r)$ is then obtained by integrating (3.21) and using toroidal flux conservation

$$\int_0^a \dot{g}_2 r \, dr \equiv 0 \tag{3.22}$$

as a constraint to determine the integration constant. Thus,

$$\dot{g}_2 = \int_r^a \left[\frac{\dot{p}_2'}{B_0^2} + \dot{f}_1 \frac{(rf_1)'}{r} + \frac{f_1}{r}(r\dot{f}_1)'\right] dr - \int_0^a \frac{r^2}{a^2}\left[\frac{\dot{p}_2'}{B_0^2} + \dot{f}_1\left(\frac{rf_1}{r}\right)' + \frac{f_1}{r}(r\dot{f}_1)'\right] dr. \tag{3.23}$$

An expression for $\dot{p}_2$ is obtained from the thermal equation (3.10). In leading order this takes the form

$$\dot{p}_2 = n_e \dot{T}_2 = \frac{1}{3}\eta_\parallel \left[\frac{B_0}{r}\frac{\partial}{\partial r}(rf_1)\right]^2 + \frac{1}{3r}\frac{\partial}{\partial r}\left(\langle\chi_\perp\rangle r n_e \frac{\partial T_2}{\partial r}\right) \tag{3.24}$$

where, within the simple model taken here with $n_e(r)$ a known stationary function, we have expanded the single-fluid temperature in the form

$$T = T_2 + T_4 \dots . \tag{3.25}$$

A higher-order equation determining $\dot{T}_4$ (and hence $\dot{p}_4$) can be obtained from (3.10), but because of the uncertainty in prescribing the form of the anomalous thermal transport $\chi_\perp(r, \theta)$ the accuracy obtained in going to this order in the thermal equation is illusory. We propose the compromise of retaining higher-order corrections in the Ohmic heating term, while neglecting all corrections to the diffusive and convective terms. The Ohmic heating term to this order is

$$P_{Oh} = \frac{2}{3}\eta_\parallel \left\{\left[\frac{B_0}{r}\frac{\partial}{\partial r}(rf_1)\right]^2 + 2\frac{B_0}{r}\frac{\partial}{\partial r}(rf_1)\left[\frac{B_0}{r}\frac{\partial}{\partial r}(rf_3) + \frac{B_0}{r}\frac{\partial}{\partial r}(rf_1\langle|\nabla r|^2\rangle_2)\right]\right.$$
$$\left. + B_0^2 g_2'^2 + \frac{2p_2'^2}{B_0^2 f_1^2}\frac{r^2}{R_0^2} - \frac{p_2'^2}{B_0^2}\right\} + \eta_\perp \frac{p_2'}{B_0^2}[\tfrac{2}{3}p_2' - n_e T_e'], \tag{3.26}$$

where $\langle|\nabla r|^2\rangle_2 \equiv (\langle|\nabla r|^2\rangle - 1)$ is again the $O(\varepsilon^2)$ part of expression (2.10).

4. BOUNDARY CONDITIONS

The boundary conditions for $f_1(r)$ and $f_3(r)$ are obtained from equations (2.7), (2.13), (3.16) and (3.19). Thus we have

$$\frac{I(t)}{2\pi a(t) B_0} = f_1 + f_3 + f_4 + f_1\left\{\langle|\nabla r|^2\rangle_2 + \langle|\nabla r|^2\rangle_3 + \left(\frac{r_a}{a} - 1\right)\right\} + O(\varepsilon^5) \tag{4.1}$$

where all quantities on the right-hand side are evaluated at $r = r_a$ and where to $O(\varepsilon^3)$

$$\left(\frac{r_a}{a}-1\right)=\left(\frac{1}{8}\frac{a^2}{R_0^2}-\frac{1}{2}\frac{S_2^2}{a^2}-\frac{S_3^2}{a^2}\right)-\frac{S_2}{4a}\left(\frac{2S_3}{R_0}+\frac{a^2}{R_0^2}\right). \tag{4.2}$$

Thus we may take

$$f_1(r_a,t)=\frac{I}{2\pi a B_0}(t) \tag{4.3}$$

describing current ramping $I(t)$, growth of the plasma $a(t)$ or steady conditions. Then

$$\hat{f}_3(r_a,t)=\hat{f}_3(r_a,0)+\Gamma(0)-\Gamma(t) \tag{4.4}$$

where

$$\Gamma(t)=f_1(r_a,t)\left[\langle|\nabla r|^2\rangle_2+\langle|\nabla r|^2\rangle_3+\left(\frac{r_a}{a}-1\right)\right]. \tag{4.5}$$

In equations (4.4) and (4.5) we have made provision for externally programming the ellipticity and triangularity of the plasma boundary.

As discussed in the previous section, for equilibrium in a perfectly-conducting shell the appropriate boundary condition for $\dot{g}_2$ is obtained from toroidal flux conservation

$$\int_0^a \dot{g}_2 r\,\mathrm{d}r=0. \tag{4.6}$$

For the case of a growing plasma this must be replaced by the condition of pressure balance at the plasma–vacuum interface. Finally, the boundary condition for the pressure can be taken as

$$p_2(a,t)=p_2(a,0), \tag{4.7}$$

i.e. a prescribed pedestal for $p_2(r)$.

5. CONCLUSIONS AND DISCUSSION

Equations (3.9) and (3.10) form the basis of the full $1\frac{1}{2}$-D description of resistive evolution in an axisymmetric plasma, wherein all explicit velocity terms have been eliminated by taking flux-surface averages. We have demonstrated that using the well-established Tokamak equilibrium formulation of GREENE *et al.* (1971) this system of equations can be expanded to high order in inverse aspect ratio. When this is done the flux-surface averages are performed analytically and there results a more tractable sequence of 1-D evolution equations [equations (3.13), (3.17), (3.24) ... etc.] that provide the basis of a very fast computational scheme. The effects of toroidicity and plasma shaping (ellipticity and triangularity) are included self-consistently at each order. To lowest order, naturally, the evolution is governed by diffusion in a straight cylinder [equation (3.12)]; the next and higher-order corrections are driven by this basic evolution.

Applications of this model currently under investigation include: (a) Evolution during the ramp phase of a Tokamak sawtooth. The use of the equilibrium formulation of GREENE *et al.* (1971) means that as well as monitoring localized stability criteria (see Appendix B), the model is well suited to follow the global toroidal stability of the plasma (MARTIN *et al.*, 1991). (b) Evolution during the initial current ramp phase of a Tokamak discharge; here the equations are generalized to permit the experimental procedure of programming of the minor and major radii to be modelled.

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APPENDIX A

The (r, θ, ϕ) coordinates are related to the cylindrical coordinates, R, Z, ϕ (with Z in the direction of the symmetry axis) by the relations

$$R = R_0 - r\cos\omega - \Delta + P\cos\omega + \sum_n S_n \cos(n-1)\omega, \tag{A1}$$

$$Z = r\sin\omega - P\sin\omega + \sum_n S_n \sin(n-1)\omega, \tag{A2}$$

where ω is the poloidal angle about the magnetic axis ($r = 0$), P a small $[O(\varepsilon^2 r)]$ departure of the "radial" variable r from the true radius of the (nearly) circular surfaces, and the $S_n(r)$ paramaterize the shape of the surfaces, with $n = 2$ representing elliptic, and $n = 3$ triangular, distortions. The angle-like variable θ is related to ω, with

$$\theta = 2\pi \int_0^{\omega} \frac{J\,\mathrm{d}\omega}{R} \Big/ \oint \frac{J\,\mathrm{d}\omega}{R}, \tag{A3}$$

where the Jacobean $J = \left(\frac{\partial R}{\partial \omega}\frac{\partial Z}{\partial r} - \frac{\partial R}{\partial r}\frac{\partial Z}{\partial \omega}\right)$ and has the property that the field line equation takes the simple form

$$\frac{\mathrm{d}\phi}{\mathrm{d}\theta} = q(r). \tag{A4}$$

The $1\frac{1}{2}$-D equations of the text require a knowledge of the time variation of $r = r(R, Z, t)$. To obtain this we differentiate equation (A1) and (A2) holding R and Z constant. We then eliminate $(\partial\omega/\partial t)_{R,Z}$ to obtain the result

$$\left(\frac{\partial r}{\partial t}\right)_{R,Z} \left\{1 - \Delta' \cos\omega - \frac{1}{r}(rP)' - r\left(\frac{S_n}{r}\right)' \cos n\omega\right\} = -\dot{\Delta}\cos\omega + \dot{P} + \sum \dot{S}_n \cos n\omega, \tag{A5}$$

where the notation $\dot{X} \equiv (\partial X(r,t)/\partial t)|_r$ has been introduced. In equation (A5) we now transform from the poloidal angle ω to the variable θ [equation (A3)], where in the leading orders

$$\omega = \theta - \left(\Delta' + \frac{r}{R_0}\right)\sin\theta + \sum_n \frac{1}{n}\left(S_n' - (n-1)\frac{S_n}{r}\right)\sin n\theta + O(\varepsilon^2), \tag{A6}$$

to obtain the required result:

$$\begin{aligned}\frac{\partial r}{\partial t} = &-\dot{\Delta}\cos\theta + \sum_n \dot{S}_n \cos n\theta + \dot{\Delta}\left(\Delta' + \frac{1}{2}\frac{r}{R_0}\right)\cos 2\theta \\ &+ \frac{1}{2}\sum_{m\neq n}\sum_n \frac{\dot{S}_n}{m}\cos(n-m)\theta\left[(m-n)S_m' + (m+n)(m-1)\frac{S_m}{r}\right] \\ &+ \frac{1}{2}\sum_m\sum_n (m+n)\frac{\dot{S}_n}{m}\cos(m+n)\theta\left[S_m' - (m-1)\frac{S_m}{r}\right] \\ &+ \frac{1}{2}\sum_n \cos(n-1)\theta\left\{\dot{S}_n\left[(n-1)\Delta' + n\frac{r}{R_0}\right] - \frac{(n-1)}{n}\dot{\Delta}\left[S_n' + (n+1)\frac{S_n}{r}\right]\right\} \\ &+ \frac{1}{2}\sum_n \cos(n+1)\theta\left\{\dot{S}_n\left[(n+1)\Delta' + n\frac{r}{R_0}\right] + \frac{(n+1)}{n}\dot{\Delta}\left[S_n' - (n-1)\frac{S_n}{r}\right]\right\}.\end{aligned} \tag{A7}$$

From (A7) we obtain the results [correct to $O(\varepsilon^2)$] required in expressions (3.9) and (3.10) of the text

$$\left\langle\frac{\partial r}{\partial t}\right\rangle = 0, \tag{A8}$$

$$\left\langle\frac{R^2}{R_0^2}\frac{\partial r}{\partial t}\right\rangle = \frac{r}{R_0}\dot{\Delta}. \tag{A9}$$

APPENDIX B

The stability criteria of MERCIER (1960) and GLASSER et al. (1975) for ideal and resistive-interchange modes, respectively, take the well-known forms

$$\frac{1}{4}\left(\frac{rq'}{q}\right)^2 + \frac{2rp'}{B_0^2}(1-q^2) > 0 \tag{B1}$$

and

$$\frac{2rp'}{B_0^2}\left(1 - q^2 - \left(\frac{q'}{q}\right)q^2 R_0\Delta'\right) > 0 \tag{B2}$$

for a circular cross-section, large aspect ratio, plasma with $\beta \sim O(\varepsilon^2)$. For a weakly-shaped Tokamak equilibrium these results remain valid in leading order, but corrections due to shaping of order $(\varepsilon^2(S_2/r))$ and $(\varepsilon(S_2S_3/r^2))$ could be important near the $q = 1$ surface, particularly if the shear is weak there. These shaping corrections have been calculated, with the aid of REDUCE (HEARN, 1987), for arbitrary equilibrium profiles of $p(r)$ and $q(r)$.

Starting from the general forms (using the notation of GLASSER et al., 1975)

$$\tfrac{1}{4} + E + F + H > 0 \tag{B3}$$

$$E + F + H^2 > 0 \tag{B4}$$

we have evaluated E, F and H to $O(\varepsilon^3)$, where

$$E + F = \left(\frac{rp'q^2}{B_0}\right)^2 \frac{R^4}{s^2 r^4}\{I_1 I_3 - I_4^2\} - \left(\frac{rp'q^2}{B_0^2}\right)\frac{1}{s^2}\left\{\frac{2}{q^2} + \frac{R_0^2}{r}\frac{\mathrm{d}I_2}{\mathrm{d}r}\right\}$$

$$H = \frac{rp'q^2}{B_0^2 s}\frac{R^2}{r^2}\{I_4 - I_1 I_2\}$$

where $s = rq'/q$ is the shear, and

$$I_1 = \left\langle \frac{1}{|\nabla r|^2} \right\rangle, \quad I_2 = \left\langle \frac{R^2}{R_0^2} \right\rangle, \quad I_3 = \left\langle \frac{R^4}{R_0^4}\frac{1}{|\nabla r|^2} \right\rangle, \quad I_4 = \left\langle \frac{R^2}{R_0^2}\frac{1}{|\nabla r|^2} \right\rangle.$$

The Mercier criterion is now found to take the form

$$\tfrac{1}{4} + D_I > 0$$

where

$$D_I = -\frac{2Rp'q^2}{s^2 B_0^2}\left\{\frac{r}{R}\left(1 - \frac{1}{q^2}\right) + \frac{3}{2}\frac{Rp'q^2}{B_0^2}\left(E' + \frac{E}{r}\right) + 3\Delta'\left(E' - \frac{E}{r} - \tfrac{1}{2}sE'\right) - \frac{1}{4}\frac{r}{R}\left(3\frac{E}{r} + 5E'\right) + (9-s)\frac{ET}{r^2} - \left(2 - \frac{3}{2}s\right)E'T' + 11\frac{E'T}{r} + 6\frac{ET'}{r}\right\}$$

where we have denoted the ellipticity $S_2(r)$ by $E(r)$, and the triangularity $S_3(r)$ by $T(r)$.

The resistive-interchange criterion of GLASSER *et al.* (1975) takes the form

$$D_R > 0$$

where

$$D_R = D_I - \frac{2Rp'}{B_0^2 s}\left\{\Delta' + \Delta_2' - \frac{3}{2}\Delta' E' + \frac{3}{2}E'T' - \frac{1}{4}E'\frac{r}{R} - \frac{E'T}{r} - \frac{T'E}{r} - \frac{ET}{r^2}\right\}.$$

Apart from their possible importance for ideal and resistive-interchange instabilities in the plasma core, where $q(r) \lesssim 1$, the quantity D_R is also of importance for tearing modes, where its value at the singular layer can influence both the linear stability threshold and non-linear island growth (KOTSCHENREUTHER *et al.*, 1985). Typically D_R is dominated by the $(1-(1/q^2))$ factor and is positive for low-m tearing modes. However, at the $q = 1$ surface this term vanishes and shaping may determine the sign of D_R. To illustrate this, we consider the value of D_R at $q = 1$, with the additional assumption that the shear there is locally very weak [$s(r_1) \ll 1$, where $q(r_1) = 1$].

Then,

$$D_R = -\frac{2Rp'}{s^2 B_0^2}\left\{\frac{3}{2}\frac{Rp'}{B_0^2}\left(E' + \frac{E}{r}\right) + 3\Delta'\left(E' - \frac{E}{r}\right) - \frac{1}{4}\frac{r}{R}\left(3\frac{E}{r} + 5E'\right) + 9\frac{ET}{r^2} - 2E'T' + 11\frac{E'}{r}T + 6\frac{ET'}{r}\right\}.$$

In deriving this expression we have not assumed that the shear $s(r)$ is weak throughout the core plasma $(0 \leqslant r \leqslant r_1)$. However, if this assumption is made, considerable simplification results since $E' \simeq E/r$ and $T' \sim 2T/r$, so that one obtains

$$D_R = -\frac{2Rp'}{B_0^2 s^2}\cdot\frac{E}{r}\left\{3\frac{Rp'}{B_0^2} - 2\frac{r}{R_0} + 28\frac{T}{r}\right\}.$$

Assuming vertical elongation of the plasma $(E > 0)$, outward-pointing triangularity $(T > 0)$ and monotonic decreasing pressure $(p' < 0)$, D_R is consequently stabilizing at the $q = 1$ surface if the triangularity at r_1 exceeds a critical value,

$$\frac{T(r_1)}{r_1} > \frac{1}{14}\left[\frac{r_1}{R_0} - \frac{3}{2}\frac{Rp'(r_1)}{B_0^2}\right].$$

APPENDIX C

We record here the ε^2 order harmonic content (i.e. the sin $m\theta$ and cos $m\theta$, $m = 1$, 2 and 3 components) of $|\nabla r|^2$, $|\nabla r \cdot \nabla \theta|$, and R^2, as obtained with the REDUCE (HEARN, 1987) symbolic algebra program.

(1) To expression (2.10) for $|\nabla r|^2$ we have to add

$$\lambda_1 \cos\theta + \lambda_2 \cos 2\theta + \lambda_3 \cos 3\theta,$$

where

$$\lambda_1 = -\frac{1}{2}\Delta' S_2' - \frac{3}{2}\frac{\Delta' S_2}{r} - 2\Delta_2' + \frac{5}{6}S_2' S_3' + \frac{2S_2' r}{R_0} + \frac{10}{3}\frac{S_2' S_3}{r} + \frac{5}{2}\frac{S_2 S_3'}{r} + \frac{2S_2 S_3}{r^2},$$

$$\lambda_2 = \frac{5}{2}\Delta'^2 + \frac{1}{3}\Delta' S_3' + \frac{\Delta' r}{R_0} - \frac{8}{3}\frac{\Delta' S_3}{r} + 2S_{22}' + \frac{3S_3' r}{R_0},$$

and

$$\lambda_3 = -\frac{11}{2}\Delta' S_2' + \frac{3}{2}\frac{\Delta' S_2}{r} - \frac{2S_2' r}{R_0} + 2S_{32}'.$$

(2) To the expression (2.11) for $\nabla r \cdot \nabla \theta$ we must add

$$\frac{1}{r}[\lambda_1 \sin\theta + \lambda_2 \sin 2\theta + \lambda_3 \sin 3\theta],$$

where

$$\lambda_1 = -\frac{3}{2}\Delta' S_2' s + \frac{5}{2}\Delta' S_2' + \frac{3}{2}\frac{\Delta' S_2}{r}s - \frac{9}{2}\frac{\Delta' S_2}{r} + \Delta_2'' r + \Delta_2' + \frac{1}{2}S_2' S_3' s - \frac{7}{2}S_2' S_3' + \frac{3}{2}\frac{S_2' p_2' q^2 R_0}{B_0^2}$$
$$- \frac{2S_2' rs}{R_0} - \frac{1}{2}\frac{S_2' r}{R_0} - \frac{10}{3}\frac{S_2' S_3}{r}s + \frac{8S_2' S_3}{r} - \frac{5}{2}\frac{S_2 S_3'}{r}s + \frac{11}{2}\frac{S_2 S_3'}{r} - \frac{3}{2}\frac{S_2 p_2' q^2 R_0}{rB_0^2} - \frac{9}{2}\frac{S_2}{R_0} - \frac{16 S_2 S_3}{r^2},$$

$$\lambda_2 = -3\Delta'^2 s + \frac{7}{2}\Delta'^2 - 2\Delta' S_3' s + 3\frac{\Delta' p_2' q^2 R_0}{B_0^2} + 3\Delta' S_3' - \frac{1}{2}\frac{\Delta' rs}{R_0} - 2\frac{\Delta' r}{R_0} + \frac{4}{3}\frac{\Delta' S_3 s}{r} - 8\Delta'\frac{S_3}{r} - \frac{1}{2}S_{22}'' r$$
$$- \frac{1}{2}S_{22}' + \frac{5}{3}\frac{S_3' p_2' q^2 R_0}{B_0^2} - \frac{3}{2}\frac{S_3' r}{R_0}s - \frac{2}{3}\frac{S_3' r}{R_0} - \frac{3}{2}\frac{S_{22}}{r} + \frac{1}{2}\frac{p_2' q^2 r}{B_0^2} - \frac{4}{3}\frac{S_3 p_2' q^2 R_0}{rB_0^2} - \frac{1}{4}\frac{r^2}{R_0^2} - \frac{32}{3}\frac{S_3}{R_0},$$

and

$$\lambda_3 = \frac{9}{2}\Delta' S_2' s - \frac{11}{2}\Delta' S_2' - \frac{1}{2}\frac{\Delta' S_2}{r}s + \frac{11}{2}\frac{\Delta' S_2}{r} - \frac{5}{2}\frac{S_2' p_2' q^2 R_0}{B_0^2} + \frac{2}{3}\frac{S_2' r}{R_0}s + \frac{3}{2}\frac{S_2' r}{R_0} - \frac{1}{3}S_{32}'' r$$
$$- \frac{1}{3}S_{32}' + \frac{1}{2}\frac{S_2}{r}\frac{p_2' q^2 R_0}{B_0^2} + \frac{3}{2}\frac{S_2}{R_0} - \frac{8}{3}\frac{S_{32}}{r}.$$

with $s = rq'/q$.

(3) To the expression (2.12) for R^2/R_0^2 we add

$$\lambda_1 \cos\theta + \lambda_2 \cos 2\theta + \lambda_3 \cos 3\theta,$$

where

$$\lambda_1 = \frac{1}{2}\frac{S_2' r}{R_0} + \frac{3}{2}\frac{S_2}{R_0}, \quad \lambda_2 = \frac{\Delta' r}{R_0} + \frac{1}{3}\frac{S_3' r}{R_0} + \frac{3}{2}\frac{r^2}{R_0^2} + \frac{4}{3}\frac{S_3}{R_0}, \quad \text{and} \quad \lambda_3 = -\frac{1}{2}\frac{S_2' r}{R_0} + \frac{1}{2}\frac{S_2}{R_0}.$$